

AN EXAMPLE OF ORTHOGONAL TRIPLE FLAG VARIETY OF FINITE TYPE

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ABSTRACT. Let G be the split special orthogonal group of degree $2n + 1$ over a field \mathbb{F} of $\text{char } \mathbb{F} \neq 2$. Then we describe G -orbits on the triple flag varieties $G/P \times G/P \times G/P$ and $G/P \times G/P \times G/B$ with respect to the diagonal action of G where P is a maximal parabolic subgroup of G of the shape $(n, 1, n)$ and B is a Borel subgroup. As by-products, we also describe GL_n -orbits on G/B , Q_{2n} -orbits on the full flag variety of GL_{2n} where Q_{2n} is the fixed-point subgroup in Sp_{2n} of a nonzero vector in \mathbb{F}^{2n} and $1 \times \text{Sp}_{2n}$ -orbits on the full flag variety of GL_{2n+1} . In the same way, we can also solve the same problem for SO_{2n} where the maximal parabolic subgroup P is of the shape (n, n) .

1. INTRODUCTION

Let G be a reductive algebraic group over a field \mathbb{F} and let P_1, \dots, P_k be parabolic subgroups of G . Then we consider the diagonal action of G on the multiple flag variety

$$\mathcal{M} = (G/P_1) \times \cdots \times (G/P_k).$$

We say \mathcal{M} is of finite type if it has finite number of G -orbits when the field \mathbb{F} is infinite.

In [MWZ99], Magyar, Weyman and Zelevinsky classified multiple flag varieties of finite type for $\text{GL}_n(\mathbb{F})$ with an arbitrary algebraically closed field \mathbb{F} and described their orbit decompositions using quiver theory. In [MWZ00], they also solved the same problem for $\text{Sp}_{2n}(\mathbb{F})$.

Consider a triple flag variety $\mathcal{M} = (G/P_1) \times (G/P_2) \times (G/P_3)$ and note that G -orbit decomposition on \mathcal{M} is naturally identified with P_3 -orbit decomposition on the double flag variety $\mathcal{D} = (G/P_1) \times (G/P_2)$. Littelmann ([L94]) classified double flag varieties \mathcal{D} with open B -orbits for simple algebraic groups G . Here P_1 and P_2 are maximal parabolic subgroups of G and B a Borel subgroup of G . Suppose that \mathcal{D} has an open B -orbit and that \mathbb{F} is an algebraically closed field of $\text{char } \mathbb{F} = 0$. Then it follows from the theorem by Brion ([B86]) and Vinberg ([V86]) that $|B \backslash \mathcal{D}|$ is finite.

We can see there are many open problems in this subject. One of them is an explicit description of orbit decomposition for each triple flag variety classified in [L94] (Table I). In this paper, we solve this problem for some typical orthogonal triple flag variety. It is interesting that we can describe orbits in our example over an arbitrary field of $\text{char } \mathbb{F} \neq 2$ and so we can also compute the number of elements in each orbit when \mathbb{F} is a finite field.

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Let \mathbb{F} be an arbitrary commutative field of $\text{char } \mathbb{F} \neq 2$. Let $(\ , \)$ denote the symmetric bilinear form on \mathbb{F}^{2n+1} defined by

$$(e_i, e_j) = \delta_{i, 2n-i+2}$$

for $i, j = 1, \dots, 2n+1$ where e_1, \dots, e_{2n+1} is the canonical basis of \mathbb{F}^{2n+1} . Define the special orthogonal group

$$G = \{g \in \text{SL}_{2n+1}(\mathbb{F}) \mid (gu, gv) = (u, v) \text{ for all } u, v \in \mathbb{F}^{2n+1}\}$$

with respect to this form. Let us write $G = \text{SO}_{2n+1}(\mathbb{F})$ in this paper. Let M denote the variety consisting of all the maximal isotropic subspaces in \mathbb{F}^{2n+1} . Here a subspace V in \mathbb{F}^{2n+1} is called a maximal isotropic subspace if $\dim V = n$ and $(V, V) = \{0\}$. Then M is a homogeneous space of G and hence it is written as $M \cong G/P$ where $P = \{g \in G \mid gU_0 = U_0\}$ with $U_0 = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_n \in M$ is a maximal parabolic subgroup of G .

Let $M_0 = \{V_1 \subset \dots \subset V_n \mid (V_n, V_n) = \{0\}\} \cong G/B$ denote the full flag variety of G . Here B is the isotropy subgroup of the canonical full flag $\mathbb{F}e_1 \subset \mathbb{F}e_1 \oplus \mathbb{F}e_2 \subset \dots \subset \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_n$ in M_0 which is called a Borel subgroup of G . In this paper, we will describe G -orbits on $\mathcal{T} = M \times M \times M$ and $\mathcal{T}_0 = M \times M \times M_0$ with respect to the diagonal action.

In the same way, we can also solve the problem for $\text{SO}_{2n}(\mathbb{F})$ (Section 1.5).

Remark 1.1. (i) Similar problems were studied in [KS90], [FMS04] and [CN06]. In particular, [KS90] (p.492) described $\text{Sp}_{2n}(\mathbb{R})$ -orbits on the variety consisting of triples of Lagrangian subspaces in a real symplectic vector space. For each orbit in this decomposition, there corresponds a symmetric bilinear form and the ‘‘Maslov index’’ is naturally defined. So it is natural that there appear alternating forms in our results on $\text{SO}_{2n+1}(\mathbb{F})$ -orbit decompositions of \mathcal{T} (Theorem 1.4) and \mathcal{T}_0 (Theorem 1.8).

(ii) We may consider the action of the orthogonal group

$$\tilde{G} = \text{O}_{2n+1}(\mathbb{F}) = \{g \in \text{GL}_{2n+1}(\mathbb{F}) \mid (gu, gv) = (u, v) \text{ for all } u, v \in \mathbb{F}^{2n+1}\}$$

on M and M_0 . But since $\tilde{G} = G \sqcup \{-g \mid g \in G\}$ and since $-I_{2n+1}$ acts trivially on M and M_0 , the \tilde{G} -orbits are the same as the G -orbits.

(iii) The triple flag variety \mathcal{T}_0 has the maximum dimension among the triple flag varieties of $\text{SO}_{2n+1}(\mathbb{F})$ of finite type since

$$\dim \mathcal{T}_0 = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + n^2 = n(2n+1) = \dim \text{SO}_{2n+1}(\mathbb{F}).$$

1.1. G -orbits on $\mathcal{T} = M \times M \times M$. For $d = 0, \dots, n$, define $U_d = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_{n-d} \oplus \mathbb{F}e_{n+2} \oplus \dots \oplus \mathbb{F}e_{n+d+1} \in M$. For a partition $n = a + b + c_+ + c_0 + c_-$ of n with nonnegative integers a, b, c_+, c_0 and c_- , define subspaces

$$\begin{aligned} U_{(\alpha)} &= \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_a, & U_{(\beta)} &= \mathbb{F}e_{2n-a-b+2} \oplus \dots \oplus \mathbb{F}e_{2n-a+1}, \\ U_{(+)} &= \mathbb{F}e_{a+b+1} \oplus \dots \oplus \mathbb{F}e_{a+b+c_+}, & U_{(-)} &= \mathbb{F}e_{n+2} \oplus \dots \oplus \mathbb{F}e_{n+c_-+1}, \\ U_{(0)} &= \mathbb{F}e_{a+b+c_++1} \oplus \dots \oplus \mathbb{F}e_{a+b+c_++c_0} \oplus \mathbb{F}e_{n+c_-+2} \oplus \dots \oplus \mathbb{F}e_{n+c_-+c_0+1} \oplus \mathbb{F}e_{n+1} \end{aligned}$$

of \mathbb{F}^{2n+1} . Write $W_{(0)} = U_{(\alpha)} \oplus U_{(\beta)} \oplus U_{(+)} \oplus U_{(-)}$, $k_+ = a + b + c_+$ and $k_- = n + c_- + 1$. If $c_0 = 2c_1 - 1$ is odd, then we define an element of M by

$$V(a, b, c_+, c_-)_{\text{odd}} = W_{(0)} \oplus \left(\bigoplus_{i=1}^{c_1-1} \mathbb{F}(e_{k_++i} + e_{k_-+i}) \right) \oplus \left(\bigoplus_{i=c_1+1}^{c_0} \mathbb{F}(e_{k_++i} - e_{k_-+i}) \right) \\ \oplus \mathbb{F}(e_{k_++c_1} - \frac{1}{2}e_{k_-+c_1} + e_{n+1}).$$

If $c_0 = 2c_1$ is even, then we define an element of M by

$$V(a, b, c_+, c_-)_{\text{even}}^0 = W_{(0)} \oplus \left(\bigoplus_{i=1}^{c_1} \mathbb{F}(e_{k_++i} + e_{k_-+i}) \right) \oplus \left(\bigoplus_{i=c_1+1}^{c_0} \mathbb{F}(e_{k_++i} - e_{k_-+i}) \right).$$

If $c_0 = 2c_1$ is even and positive, then we also define

$$V(a, b, c_+, c_-)_{\text{even}}^1 = W_{(0)} \oplus \left(\bigoplus_{i=1}^{c_1} \mathbb{F}(e_{k_++i} + e_{k_-+i}) \right) \oplus \left(\bigoplus_{i=c_1+1}^{c_0-1} \mathbb{F}(e_{k_++i} - e_{k_-+i}) \right) \\ \oplus \mathbb{F}(e_{k_++c_0} - e_{k_-+c_0} - \frac{1}{2}e_{k_-+1} + e_{n+1}) \in M.$$

Theorem 1.2. Let $t = (V_{(1)}, V_{(2)}, V_{(3)})$ be an element of $\mathcal{T} = M \times M \times M$. Define

$$a = a(t) = \dim(V_{(1)} \cap V_{(2)} \cap V_{(3)}), \quad b = b(t) = \dim(V_{(1)} \cap V_{(2)}) - a, \\ c_+ = c_+(t) = \dim(V_{(1)} \cap V_{(3)}) - a, \quad c_- = c_-(t) = \dim(V_{(2)} \cap V_{(3)}) - a, \\ c_0 = c_0(t) = n - a - b - c_+ - c_-$$

$$\text{and } \varepsilon = \varepsilon(t) = \dim(V_{(1)} + V_{(2)} + V_{(3)}) + \dim(V_{(1)} \cap V_{(2)} \cap V_{(3)}) - 2n \in \{0, 1\}.$$

- (i) If c_0 is odd, then $\varepsilon = 1$ and $t \in G(U_0, U_{n-a-b}, V(a, b, c_+, c_-)_{\text{odd}})$.
- (ii) If $c_0 = 0$, then $\varepsilon = 0$ and $t \in G(U_0, U_{n-a-b}, V(a, b, c_+, c_-)_{\text{even}}^0)$.
- (iii) If c_0 is even and positive, then $t \in G(U_0, U_{n-a-b}, V(a, b, c_+, c_-)_{\text{even}}^\varepsilon)$ with $\varepsilon = 0$ or 1 .

Corollary 1.3. $|G \backslash \mathcal{T}| = \sum_{k=0}^n \eta_k \binom{n-k+3}{3}$ where $\eta_k = \begin{cases} 1 & \text{if } k = 0, 1, 3, 5, \dots, \\ 2 & \text{if } k = 2, 4, 6, \dots \end{cases}$

For $n = 1, 2, 3, 4$, the number of orbits $|G \backslash \mathcal{T}|$ is as follows.

n	1	2	3	4
$ G \backslash \mathcal{T} $	5	16	39	81

Theorem 1.4. When \mathbb{F} is the finite field \mathbb{F}_r with r elements, the number of elements in the G -orbit Gt is

$$|Gt| = |M| \frac{r^{(n-a)(n-a+1)/2} [r]_n}{[r]_a [r]_b [r]_{c_+} [r]_{c_-} [r]_{c_0}} \psi_{c_0}^\varepsilon(r).$$

Here $[r]_m$ is the r -factorial number $(r+1)(r^2+r+1)\cdots(r^{m-1}+r^{m-2}+\cdots+1)$ and

$$\psi_{2k}^0(r) = \psi_{2k-1}^1(r) = \frac{\psi_{2k}^1(r)}{r^{2k}-1} = r^{k(k-1)}(r-1)(r^3-1)\cdots(r^{2k-1}-1).$$

Remark 1.5. (c.f. Proposition 1.7) $\psi_{c_0}^\varepsilon(r) = |\mathrm{GL}_{c_0}(\mathbb{F}_r)/H_{c_0}^\varepsilon|$ where

$$H_{c_0}^\varepsilon = \begin{cases} 1 \times \mathrm{Sp}_{c_0-1}(\mathbb{F}_r) & \text{if } c_0 \text{ is odd,} \\ \mathrm{Sp}_{c_0}(\mathbb{F}_r) & \text{if } c_0 \text{ is even and } \varepsilon = 0, \\ Q_{c_0} = \{g \in \mathrm{Sp}_{c_0}(\mathbb{F}_r) \mid gv = v\} & \text{if } c_0 \text{ is even and } \varepsilon = 1. \end{cases}$$

(v is a nonzero element in $\mathbb{F}_r^{c_0}$.)

1.2. G -orbits on $\mathcal{T}_0 = M \times M \times M_0$. By Theorem 1.2, we may fix a $t = (U_0, U_d, V)$ with $V \in M$ of the form

$$V = V(a, b, c_+, c_-)_{\text{odd}}, \quad V(a, b, c_+, c_-)_{\text{even}}^0 \quad \text{or} \quad V(a, b, c_+, c_-)_{\text{even}}^1$$

where $d = n - a - b$. Let $M_0(V)$ denote the subvariety of M_0 consisting of full flags $\mathcal{F} : V_1 \subset \cdots \subset V_n$ satisfying $V_n = V$. Let $\pi : \mathcal{T}_0 \rightarrow \mathcal{T}$ be the projection. Then the fiber $\pi^{-1}(t)$ at t is naturally identified with $M_0(V)$. Since the isotropy subgroup at t is $R(t) = P \cap P_{U_d} \cap P_V$. We have only to describe $R(t)$ -orbits on $M_0(V)$.

Definition 1.6. A full flag $\mathcal{F} : V_1 \subset \cdots \subset V_n$ in $M_0(V)$ is called standard if

$$V_i = (V_i \cap U_{(\alpha)}) \oplus (V_i \cap U_{(\beta)}) \oplus (V_i \cap (U_{(+)} \oplus U_{(-)})) \oplus (V_i \cap U_{(0)}),$$

$V_i \cap U_{(\alpha)} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_i(\mathcal{F})}$ and $V_i \cap U_{(\beta)} = \mathbb{F}e_{2n-a-b+2} \oplus \cdots \oplus \mathbb{F}e_{2n-a-b+1+b_i(\mathcal{F})}$ for all $i = 1, \dots, n$ where $a_i(\mathcal{F}) = \dim(V_i \cap U_{(\alpha)})$ and $b_i(\mathcal{F}) = \dim(V_i \cap U_{(\beta)})$.

For a standard full flag $\mathcal{F} : V_1 \subset \cdots \subset V_n$, write $c_i(\mathcal{F}) = \dim(V_i \cap (U_{(+)} \oplus U_{(-)}))$ and $d_i(\mathcal{F}) = \dim(V_i \cap U_{(0)})$. Define subsets

$$\begin{aligned} I_{(\alpha)} &= \{\alpha_1, \dots, \alpha_a\} = \{i \in I \mid a_i(\mathcal{F}) = a_{i-1}(\mathcal{F}) + 1\}, \\ I_{(\beta)} &= \{\beta_1, \dots, \beta_b\} = \{i \in I \mid b_i(\mathcal{F}) = b_{i-1}(\mathcal{F}) + 1\}, \\ I_{(\gamma)} &= \{\gamma_1, \dots, \gamma_c\} = \{i \in I \mid c_i(\mathcal{F}) = c_{i-1}(\mathcal{F}) + 1\}, \\ I_{(\delta)} &= \{\delta_1, \dots, \delta_{c_0}\} = \{i \in I \mid d_i(\mathcal{F}) = d_{i-1}(\mathcal{F}) + 1\} \end{aligned}$$

of $I = \{1, \dots, n\}$ where $c = c_+ + c_-$ and $\alpha_1 < \cdots < \alpha_a$, $\beta_1 < \cdots < \beta_b$, $\gamma_1 < \cdots < \gamma_c$, $\delta_1 < \cdots < \delta_{c_0}$. Let $\tau(\mathcal{F})$ denote the permutation

$$\tau(\mathcal{F}) : (1 \ 2 \ \cdots \ n) \mapsto (\alpha_1 \cdots \alpha_a \gamma_1 \cdots \gamma_c \delta_1 \cdots \delta_{c_0} \beta_1 \cdots \beta_b)$$

of I and $\ell(\tau(\mathcal{F}))$ the inversion number of $\tau(\mathcal{F})$.

For $X \in \mathrm{GL}_n(\mathbb{F})$, write

$$h[X] = \begin{pmatrix} X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & J^t X^{-1} J \end{pmatrix} \quad \text{with } J = J_n = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

For $A \in \mathrm{GL}_{c_+}(\mathbb{F})$, $B \in \mathrm{GL}_{c_0}(\mathbb{F})$ and $C \in \mathrm{GL}_{c_-}(\mathbb{F})$, define an element

$$\ell(A, B, C) = h \left[\begin{pmatrix} I_{a+b} & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \right]$$

of G . Let L_+ , L_0 , L_- , L and L_V denote the subgroups of G defined by

$$\begin{aligned} L_+ &= \{\ell(A, I_{c_0}, I_{c_-}) \mid A \in \mathrm{GL}_{c_+}(\mathbb{F})\}, \\ L_0 &= \{\ell(I_{c_+}, B, I_{c_-}) \mid B \in \mathrm{GL}_{c_0}(\mathbb{F})\}, \\ L_- &= \{\ell(I_{c_+}, I_{c_0}, C) \mid C \in \mathrm{GL}_{c_-}(\mathbb{F})\}, \end{aligned}$$

$L = L_+ \times L_0 \times L_-$ and $L_V = \{\ell \in L \mid \ell V = V\}$, respectively.

Proposition 1.7. (i) $L_V = L_+ \times (L_V \cap L_0) \times L_-$.

(ii) $V = V(a, b, c_+, c_-)_{\text{odd}} \implies L_V \cap L_0 \cong 1 \times \mathrm{Sp}_{c_0-1}(\mathbb{F})$,

$V = V(a, b, c_+, c_-)_{\text{even}}^0 \implies L_V \cap L_0 \cong \mathrm{Sp}_{c_0}(\mathbb{F})$,

$V = V(a, b, c_+, c_-)_{\text{even}}^1 \implies L_V \cap L_0 \cong Q_{c_0}$.

Here $Q_{c_0} = \{g \in \mathrm{Sp}_{c_0}(\mathbb{F}) \mid gv = v\}$ with some $v \in \mathbb{F}^{c_0} - \{0\}$.

Theorem 1.8. (i) For every full flag \mathcal{F} in $M_0(V)$, there exists a $g \in R(t) = P \cap P_{U_d} \cap P_V$ such that $g\mathcal{F}$ is standard.

(ii) Let \mathcal{F} and \mathcal{F}' be two standard full flags such that $g\mathcal{F} = \mathcal{F}'$ for some $g \in R(t)$. Then there exists a $g_L \in L_V$ such that $g_L\mathcal{F} = \mathcal{F}'$.

(iii) If $\mathbb{F} = \mathbb{F}_r$, then $|R(t)\mathcal{F}| = [r]_a[r]_b r^{\ell(\tau(\mathcal{F}))} |L_V\mathcal{F}|$ for each standard full flag \mathcal{F} in $M_0(V)$.

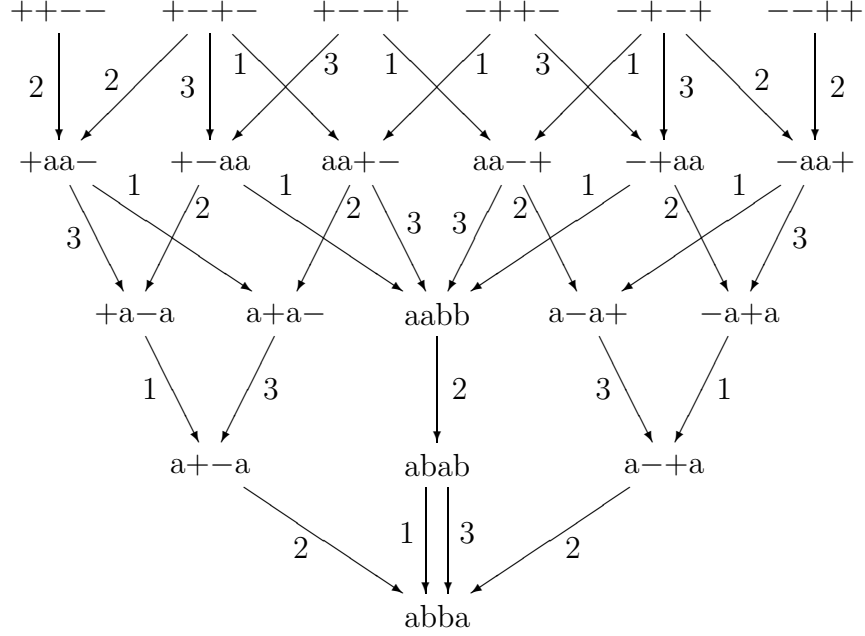
1.3. Orbits on $\mathrm{GL}_n(\mathbb{F})/B$. By Proposition 1.7 and Theorem 1.8, our problem is reduced to the orbit decompositions on the full flag variety of $\mathrm{GL}_n(\mathbb{F})$ with respect to the following four kinds of subgroups H of $\mathrm{GL}_n(\mathbb{F})$:

- (A) $H = \mathrm{GL}_{m_+}(\mathbb{F}) \times \mathrm{GL}_{m_-}(\mathbb{F})$ where $m_+ + m_- = n$,
- (B) $H = \mathrm{Sp}_n(\mathbb{F})$ for even n ,
- (C) $H = Q_n$ for even n ,
- (D) $H = 1 \times \mathrm{Sp}_{n-1}(\mathbb{F})$ for odd n .

When $\mathbb{F} = \mathbb{C}$, the subgroups H in (A) and (B) are symmetric subgroups of $\mathrm{GL}_n(\mathbb{C})$ and the orbit structures were described in [M79] and [R79]. We also have symbolic description of orbits in [MO90].

We will solve Problems (B), (C) and (D) in Section 3. We will also give a proof for the Problem (A) in the appendix. We don't need the assumption $\mathrm{char} \mathbb{F} \neq 2$ for these problems.

We can express $\mathrm{GL}_{m_+}(\mathbb{F}) \times \mathrm{GL}_{m_-}(\mathbb{F})$ -orbits on $M = \mathrm{GL}_n(\mathbb{F})/B$ by “+−ab-symbols”. For example, when $m_+ = m_- = 2$, the orbit structure is as follows (Fig.7 in [MO90]).

Fig.1. $\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_2(\mathbb{F}) \backslash \mathrm{GL}_4(\mathbb{F}) / B$

Notation: For $i = 1, \dots, n-1$, we can consider the partial flag variety

$$M_i = \{V_1 \subset \dots \subset V_{i-1} \subset V_{i+1} \subset \dots \subset V_{n-1} \mid \dim V_j = j\}$$

and the canonical projection $p_i : M \rightarrow M_i$. For two H -orbits S_1 and S_2 in M , we write $S_1 \xrightarrow{i} S_2$ when $p_i(S_1) = p_i(S_2)$ and $\dim S_1 + 1 = \dim S_2$. (Remark: In our setting, every orbit in M is defined by linear equations. So we can define “dimension” of each orbit over an arbitrary field \mathbb{F} . When $\mathbb{F} = \mathbb{F}_r$, the number $|S|$ of points in a orbit S is a polynomial of r and $\dim S$ is the degree of the polynomial.) This notation will be also used in Problems (B), (C) and (D).

Problem (B) is solved as follows. Let $G = \mathrm{GL}_{2n}(\mathbb{F})$ and define a nondegenerate alternating form $\langle \cdot, \cdot \rangle$ on \mathbb{F}^{2n} by

$$\langle e_i, e_j \rangle = \begin{cases} \delta_{i, 2n+1-j} & \text{for } i = 1, \dots, n, \\ -\delta_{i, 2n+1-j} & \text{for } i = n+1, \dots, 2n. \end{cases}$$

For a subspace V in \mathbb{F}^{2n} , let $V^\perp = \{v \in V \mid \langle v, V \rangle = \{0\}\}$ denote the orthogonal space for V . Define $H = \{g \in G \mid \langle gu, gv \rangle = \langle u, v \rangle \text{ for all } u, v \in \mathbb{F}^{2n}\}$. Then H is isomorphic to $\mathrm{Sp}_{2n}(\mathbb{F})$. Let $\mathcal{F} : V_1 \subset V_2 \subset \dots \subset V_{2n-1}$ be a full flag in \mathbb{F}^{2n} . Define $d_{i,j} = d_{i,j}(\mathcal{F}) = \dim(V_i \cap V_j^\perp)$ for $i, j = 0, \dots, 2n$ and $c_{i,j} = c_{i,j}(\mathcal{F}) = d_{i,j-1} - d_{i,j} - d_{i-1,j-1} + d_{i-1,j}$ for $i, j = 1, \dots, 2n$. We prove the following propositions in Section 3.2.

Proposition 1.9. *The matrix $\{c_{i,j}\}_{i=1,\dots,2n}^{j=1,\dots,2n}$ is a symmetric permutation matrix such that $c_{i,i} = 0$ for $i = 1, \dots, 2n$.*

Let $\tau = \tau(\mathcal{F})$ be the permutation of $I = \{1, \dots, 2n\}$ corresponding to $\{c_{i,j}\} = \{c_{i,j}(\mathcal{F})\}$. Then τ is expressed as $\tau = (i_1 j_1) \cdots (i_n j_n)$ with transpositions $(i_1 j_1), \dots, (i_n j_n)$. We may assume

$$i_t < j_t \text{ for } t = 1, \dots, n \quad \text{and} \quad i_1 < i_2 < \cdots < i_n.$$

(Hence $i_1 = 1$.) Define another permutation σ by

$$\sigma = \sigma(\mathcal{F}) : (1 \ 2 \ \cdots \ 2n) \mapsto (i_1 j_1 i_2 j_2 \cdots i_n j_n).$$

Let $\ell(\sigma)$ denote the inversion number $\ell(\sigma) = |\{(i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j)\}|$. (Remark: We can prove $\ell(\tau) = n + 2\ell(\sigma)$.)

Proposition 1.10. (i) *There exists a basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} satisfying (a) and (b):*

(a) $V_i = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_i$ for $i = 1, \dots, 2n$.

(b) $\langle v_i, v_j \rangle = c_{i,j}$ for $i < j$.

(ii) *If $\mathbb{F} = \mathbb{F}_r$, then the number of bases satisfying the properties (a) and (b) in (i) is $(r-1)^n r^{n+\ell(\sigma)}$.*

Let C_{2n} denote the set of symmetric permutation matrices $\{c_{i,j}\}$ of degree $2n$ such that $c_{i,i} = 0$ for $i = 1, \dots, 2n$. By Proposition 1.9 and Proposition 1.10, we have:

Corollary 1.11. (i) *There exists a one-to-one correspondence between C_{2n} and $\mathrm{Sp}_{2n}(\mathbb{F}) \backslash \mathrm{GL}_{2n}(\mathbb{F}) / B$.*

(ii) $|\mathrm{Sp}_{2n}(\mathbb{F}) \backslash \mathrm{GL}_{2n}(\mathbb{F}) / B| = (2n-1)(2n-3) \cdots 1 = \frac{(2n)!}{2^n n!}.$

(iii) *If $\mathbb{F} = \mathbb{F}_r$, then*

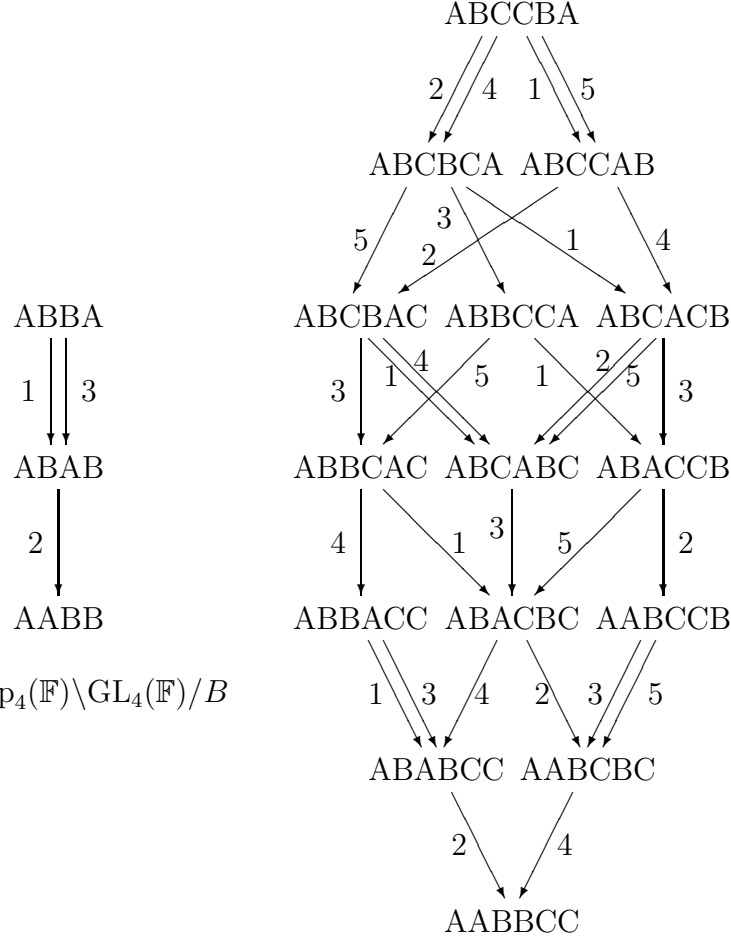
$$|H\mathcal{F}| = \frac{|\mathrm{Sp}_{2n}(\mathbb{F}_r)|}{(r-1)^n r^{n+\ell(\sigma(\mathcal{F}))}} = \frac{(r^2-1)(r^4-1) \cdots (r^{2n}-1)}{(r-1)^n} r^{n^2-n-\ell(\sigma(\mathcal{F}))}.$$

By this result, we can describe $\mathrm{Sp}_{2n}(\mathbb{F}) \backslash \mathrm{GL}_{2n}(\mathbb{F}) / B$ by the “AB-symbols”. For $n = 2, 3$, the orbit structure is as in Fig.2 and Fig.3 (Fig.3 and Fig.4 in [MO90]). For example, the symbol ABBA implies the Sp_4 -orbit of the flag \mathcal{F} such that $c_{1,4} = c_{2,3} = 1$.

Problem (C) is solved as follows. Retain the notations for Problem (B). Define a subgroup $Q_{2n} = \{g \in H \mid ge_{2n} = e_{2n}\}$ of $H \cong \mathrm{Sp}_{2n}(\mathbb{F})$. Write $W = (\mathbb{F}e_{2n})^\perp = \mathbb{F}e_2 \oplus \cdots \oplus \mathbb{F}e_{2n}$. Let S denote the subset of $I \times I$ defined by $S = S(\mathcal{F}) = \{(i, j) \mid V_i \cap V_{j-1}^\perp \not\subset W\}$ and define a subset

$$S_0 = S_0(\mathcal{F}) = \{(i, j) \in S \mid V_i \cap V_j^\perp \subset W \text{ and } V_{i-1} \cap V_{j-1}^\perp \subset W\}$$

of S . Then the following results will be proved in Section 3.3.

Fig.2. $\mathrm{Sp}_4(\mathbb{F}) \backslash \mathrm{GL}_4(\mathbb{F}) / B$ Fig.3. $\mathrm{Sp}_6(\mathbb{F}) \backslash \mathrm{GL}_6(\mathbb{F}) / B$

Proposition 1.12. (i) If $(i, j) \in S_0$ then $c_{i,j} = 1$.

(ii) We can write $S_0 = \{(x_1, y_1), (x_2, y_2), \dots, (x_s, y_s)\}$ with some $x_1 < x_2 < \dots < x_s$ and $y_1 < y_2 < \dots < y_s$ satisfying $\{x_1, \dots, x_s\} \cap \{y_1, \dots, y_s\} = \emptyset$.

Define $m = m(\mathcal{F}) = |\{(i, j) \mid c_{i,j} = 1 \text{ and } (i, j) \in S - S_0\}|$.

Proposition 1.13. (i) There exists a basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} satisfying (a), (b) and (c):

(a) $V_i = \mathbb{F}v_1 \oplus \dots \oplus \mathbb{F}v_i$ for $i = 1, \dots, 2n$.

(b) $\langle v_i, v_j \rangle = c_{i,j}$ for $i < j$.

(c) $v_i \in W$ for $i \neq x_1, \dots, x_s$ and $\langle v_{x_1}, e_{2n} \rangle = \dots = \langle v_{x_s}, e_{2n} \rangle = 1$.

(ii) If $\mathbb{F} = \mathbb{F}_r$, then the number of bases satisfying the properties (a), (b) and (c) in (i) is $(r-1)^{n-s} r^{n+\ell(\sigma)-m}$.

Theorem 1.14. (i) *There exists a one-to-one correspondence between*

$$\bigsqcup_{s=1}^n \bigsqcup_* C(I_{(A)}) \quad \text{and} \quad Q_{2n} \backslash \mathrm{GL}_{2n}(\mathbb{F})/B.$$

$$\begin{aligned} \text{(ii)} \quad & |Q_{2n} \backslash \text{GL}_{2n}(\mathbb{F})/B| = \sum_{s=1}^n \frac{(2n)!}{2^{n-s}(s!)^2(n-s)!}. \\ \text{(iii)} \quad & \text{If } \mathbb{F} = \mathbb{F}_r, \text{ then} \end{aligned}$$

$$\begin{aligned} |Q_{2n}\mathcal{F}| &= \frac{|Q_{2n}|}{(r-1)^{n-s(\mathcal{F})}r^{n+\ell(\sigma(\mathcal{F}))-m(\mathcal{F})}} \\ &= \frac{(r^2-1)(r^4-1)\cdots(r^{2n-2}-1)}{(r-1)^{n-s(\mathcal{F})}}r^{n^2-n-\ell(\sigma(\mathcal{F}))+m(\mathcal{F})}. \end{aligned}$$

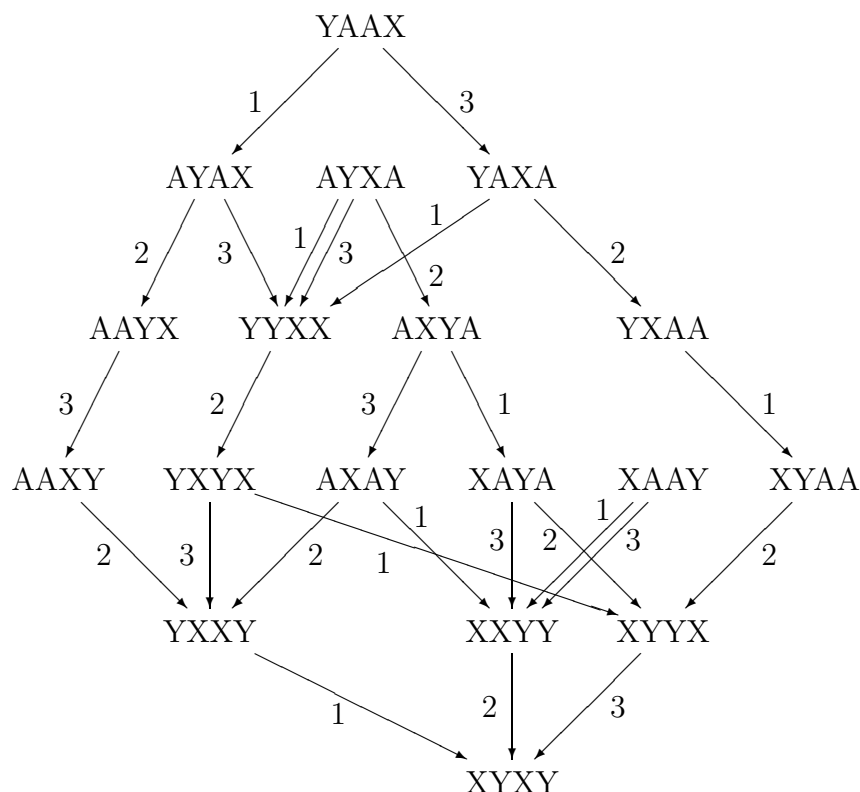


Fig.4. $Q_4 \backslash \mathrm{GL}_4(\mathbb{F}) / B$

For $n = 1, 2, 3, 4$, the number of orbits $|Q_{2n} \backslash \text{GL}_{2n}(\mathbb{F})/B|$ is as follows.

n	1	2	3	4
$ Q_{2n} \backslash \text{GL}_{2n}(\mathbb{F})/B $	2	18	200	2730

We can express orbits by “ABXY-symbols”. When $n = 2$, the orbit structure is as in Fig.4. In the diagram, the symbol XYYX implies the Q_4 -orbit containing the flag \mathcal{F} such that $S_0(\mathcal{F}) = \{(1, 2), (3, 4)\}$, for example.

Finally we will solve Problem (D) only restricting Problem (C) to a subgroup of $\text{GL}_{2n}(\mathbb{F})$. Retain the notations in Problem (C). Define a subspace $W' = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{2n-1} = (\mathbb{F}e_1)^\perp$ of \mathbb{F}^{2n} . Then $Q'_{2n} = \{g \in Q_{2n} \mid gW' = W'\}$ is written as

$$Q'_{2n} = \{g \in H \mid ge_1 = e_1 \text{ and } ge_{2n} = e_{2n}\} \cong 1 \times \text{Sp}_{2n-2}(\mathbb{F}) \subset \text{GL}(W').$$

Consider the variety M' consisting of full flags $V_1 \subset \cdots \subset V_{2n-1}$ satisfying $V_{2n-1} = W'$. Then M' is the full flag variety of $\text{GL}(W')$. Two flags in M' are in the same Q'_{2n} -orbit if and only if they are in the same Q_{2n} -orbit.

For a full flag $V_1 \subset \cdots \subset V_{2n-1}$ in M' , let i be the least integer such that $V_i \supset \mathbb{F}e_1$. Then the pair $(i, 2n)$ is contained in S_0 since $V_i \cap V_{2n-1}^\perp = \mathbb{F}e_1 \not\subset W$. This implies $x_s = i$ and $y_s = 2n$. Thus we have:

Theorem 1.15. (i) *There exists a one-to-one correspondence between*

$$\bigsqcup_{s=1}^n \bigsqcup_{*} C(I_{(A)}) \quad \text{and} \quad 1 \times \text{Sp}_{2n-2}(\mathbb{F}) \backslash \text{GL}_{2n-1}(\mathbb{F})/B.$$

Here the disjoint union $*$ is taken for all the partitions $I = \{1, \dots, 2n-1\} = I_{(A)} \sqcup I_{(X)} \sqcup I_{(Y)}$ such that $|I_{(X)}| = s$ and that $|I_{(Y)}| = s-1$.

$$(ii) \quad |1 \times \text{Sp}_{2n-2}(\mathbb{F}) \backslash \text{GL}_{2n-1}(\mathbb{F})/B| = \sum_{s=1}^n \frac{(2n-1)!}{2^{n-s}s!(s-1)!(n-s)!}.$$

(iii) *If $\mathbb{F} = \mathbb{F}_r$, then*

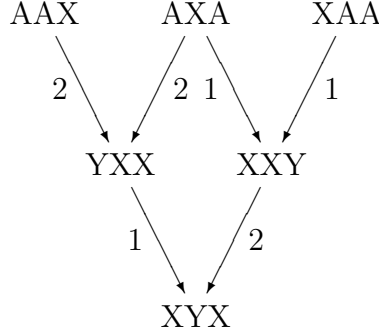
$$\begin{aligned} |(1 \times \text{Sp}_{2n-2}(\mathbb{F}))\mathcal{F}| &= \frac{|\text{Sp}_{2n-2}(\mathbb{F})|}{(r-1)^{n-s(\mathcal{F})}r^{n+\ell(\sigma(\mathcal{F}))-m(\mathcal{F})}} \\ &= \frac{(r^2-1)(r^4-1)\cdots(r^{2n-2}-1)}{(r-1)^{n-s(\mathcal{F})}} r^{(n-1)^2-n-\ell(\sigma(\mathcal{F}))+m(\mathcal{F})} \end{aligned}$$

Here the invariants $\sigma(\mathcal{F})$ and $m(\mathcal{F})$ are defined for the natural extension of the full flag \mathcal{F} to \mathbb{F}^{2n} as is explained above.

For $n \leq 5$, the number of orbits $|1 \times \text{Sp}_{2n-2}(\mathbb{F}) \backslash \text{GL}_{2n-1}(\mathbb{F})/B|$ is as follows.

n	1	2	3	4	5
$ 1 \times \text{Sp}_{2n-2}(\mathbb{F}) \backslash \text{GL}_{2n-1}(\mathbb{F})/B $	1	6	55	665	9891

We can express $1 \times \text{Sp}_2(\mathbb{F})$ -orbits on $\text{GL}_3(\mathbb{F})/B$ as follows:


 Fig.5. $1 \times \mathrm{Sp}_2(\mathbb{F}) \backslash \mathrm{GL}_3(\mathbb{F}) / B$

We have only to extract from the diagram of $Q_4 \backslash \mathrm{GL}_4(\mathbb{F}) / B$ six symbols containing the letter “Y” as the fourth letter and then delete these “Y”. Clearly the orbit structure is the same as $\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_1(\mathbb{F}) \backslash \mathrm{GL}_3(\mathbb{F}) / B$ (c.f. [MO90], Fig.5).

1.4. Expression by symbols and number of orbits. By Theorem 1.8, Corollary 1.11, Theorem 1.14, Theorem 1.15 and Proposition 4.2, we can attach each G -orbit Gt ($t = (U_0, U_d, \mathcal{F})$, $d = n - a - b$) on $\mathcal{T}_0 = M \times M \times M_0$ a “word” $w = \ell_1 \cdots \ell_n$ consisting of letters ℓ_i as follows. Write $I = \{1, \dots, n\} = I_{(\alpha)} \sqcup I_{(\beta)} \sqcup I_{(\gamma)} \sqcup I_{(\delta)}$ as in Section 1.2. Then

$$\begin{aligned} i \in I_{(\alpha)} &\implies \ell_i = \alpha, & i \in I_{(\beta)} &\implies \ell_i = \beta, & i \in I_{(\gamma)} &\implies \ell_i = +, - \text{ or } a, b, \dots, \\ i \in I_{(\delta)} &\implies \ell_i = X, Y \text{ or } A, B, \dots \end{aligned}$$

Here the subword $w_{(\gamma)} = \ell_{\gamma_1} \cdots \ell_{\gamma_c}$ expresses an $L_+ \times L_-$ -orbit of the full flag $V_{\gamma_1} \cap W^0 \subset \cdots \subset V_{\gamma_c} \cap W^0$ in $W^0 = U_{(+)} \oplus U_{(-)}$ as in Section 1.3 and Section 4. On the other hand, the subword $w_{(\delta)} = \ell_{\delta_1} \cdots \ell_{\delta_{c_0}}$ expresses an $L_V \cap L_0$ -orbit of the full flag $V_{\delta_1} \cap U_{(0)} \subset \cdots \subset V_{\delta_{c_0}} \cap U_{(0)}$ in $U_{(0)}$ as in Section 1.3.

Example 1.16. (The case of $n = 2$) We can describe $G \backslash \mathcal{T}_0 \cong \bigsqcup_{d=0}^2 R_d \backslash M_0$ as in Fig.6 when $n = 2$.

Notation: Let $p_i : M_0 \rightarrow M_i$ ($i = 1, 2$) be the canonical projections where $M_1 = \{V_2 \mid \dim V_2 = 2\}$ ($= M$) and $M_2 = \{V_1 \mid \dim V_1 = 1\}$ are partial flag varieties. Then for two R_d -orbits S_1 and S_2 on M_0 , we write $S_1 \xrightarrow{i} S_2$ when $p_i(S_1) = p_i(S_2)$ and $\dim S_1 + 1 = \dim S_2$.

Remark 1.17. Suppose $\mathbb{F} = \mathbb{C}$. Then $G = \mathrm{SO}_5(\mathbb{C}) \cong \mathrm{Sp}_4(\mathbb{C}) / \mathbb{Z}_2$ in this case. So the orbit structure is the same as the symplectic triple flag variety of the shape $(121)(121)(1^4)$ given in [MWZ00].

Using these symbols, we can easily count the number of orbits as follows. Let $\xi(k)$ denote the number of words consisting of k letters which do not contain the four letters $\alpha, \beta, +$ and $-$.

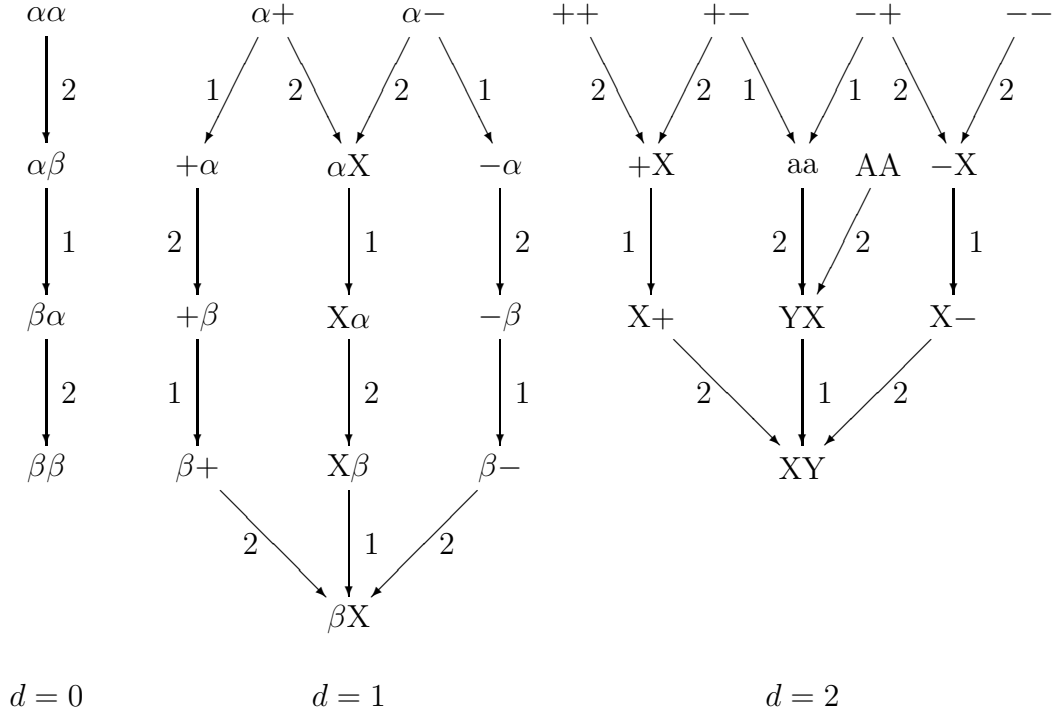


Fig.6. $\bigsqcup_{d=0}^2 R_d \backslash \mathrm{SO}_5(\mathbb{F})/B$

Lemma 1.18. $\xi(2k) = \sum_{s=0}^k \frac{(2k)!}{(s!)^2(k-s)!}$ and $\xi(2k-1) = \sum_{s=1}^k \frac{(2k-1)!}{s!(s-1)!(k-s)!}$.

Theorem 1.19. (i) $|G \backslash \mathcal{T}_0| = \sum_{k=0}^n 4^{n-k} \binom{n}{k} \xi(k)$.

(ii) $|\mathrm{GL}_n(\mathbb{F}) \backslash M_0| = \sum_{k=0}^n 2^{n-k} \binom{n}{k} \xi(k)$.

For $n = 1, 2, 3, 4$, the number of orbits is as follows.

n	1	2	3	4
$ G \backslash \mathcal{T}_0 $	5	28	169	1082
$ \mathrm{GL}_n(\mathbb{F}) \backslash M_0 $	3	12	53	258

1.5. **The case of $\mathrm{SO}_{2n}(\mathbb{F})$.** Let G' and \tilde{G}' be the subgroups of G defined by

$$G' = \{g \in G \mid ge_{n+1} = e_{n+1}\} \quad \text{and} \quad \tilde{G}' = \{g \in G \mid ge_{n+1} = \pm e_{n+1}\},$$

respectively. Then they are isomorphic to the split special orthogonal group and the split orthogonal group of degree $2n$, respectively. Let M' be the subvariety of M defined by $M' = \{V \in M \mid (V, e_{n+1}) = \{0\}\}$.

Then M' is a homogeneous space of \tilde{G}' consisting of two G' -orbits $M^0 = G'U_0$ and $M^1 = G'U_1$ ($U_1 = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{n-1} \oplus \mathbb{F}e_{n+2}$). Note that V and V' in M' are contained in the same G' -orbit if and only if $n - \dim(V \cap V')$ is even. Define a subvariety $M'_0 = \{\mathcal{F} : V_1 \subset \cdots \subset V_n \mid V_n \in M'\}$ of M_0 . Then M'_0 is also a homogeneous space of \tilde{G}' consisting of two G' -orbits $M'_0{}^0$ and $M'_0{}^1$.

Theorem 1.20. *Let $t = (V_{(1)}, V_{(2)}, V_{(3)})$ be an element of $\mathcal{T}' = M' \times M' \times M'$. Define*

$$\begin{aligned} a &= a(t) = \dim(V_{(1)} \cap V_{(2)} \cap V_{(3)}), \quad b = b(t) = \dim(V_{(1)} \cap V_{(2)}) - a, \\ c_+ &= c_+(t) = \dim(V_{(1)} \cap V_{(3)}) - a, \quad c_- = c_-(t) = \dim(V_{(2)} \cap V_{(3)}) - a, \\ c_0 &= c_0(t) = n - a - b - c_+ - c_- \\ \text{and } \varepsilon &= \varepsilon(t) = \dim(V_{(1)} + V_{(2)} + V_{(3)}) + \dim(V_{(1)} \cap V_{(2)} \cap V_{(3)}) - 2n \in \{0, 1\}. \end{aligned}$$

Then we have:

- (i) $\varepsilon = 0$, c_0 is even and $t \in \tilde{G}'(U_0, U_{n-a-b}, V(a, b, c_+, c_-)_{\text{even}}^0)$.
- (ii) If $\mathbb{F} = \mathbb{F}_r$, then $|\tilde{G}'t| = 2|G't| = |M'| \frac{r^{(n-a)(n-a-1)} [r]_n \psi_{c_0}^0(r)}{[r]_a [r]_b [r]_{c_+} [r]_{c_0} [r]_{c_-}}$.

Corollary 1.21. (i) $|\tilde{G}' \backslash \mathcal{T}'| = \frac{|G' \backslash \mathcal{T}'|}{2} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-2k+3}{3}.$

$$(ii) \quad |G' \backslash M^{\nu_1} \times M^{\nu_2} \times M^{\nu_3}| = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{k+3}{3} + \sum_{k=0}^{\lfloor (n-3)/2 \rfloor} \binom{k+3}{3} \text{ if } \nu_1 = \nu_2 = \nu_3.$$

$$(iii) \quad |G' \backslash M^{\nu_1} \times M^{\nu_2} \times M^{\nu_3}| = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{k+3}{3} + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{k+3}{3} \text{ if } \nu_i \neq \nu_j \text{ for}$$

some $i, j = 1, 2, 3$.

For $n = 2, 3, 4, 5$, the number of orbits is as follows.

n	2	3	4	5
$ \tilde{G}' \backslash \mathcal{T}' = G' \backslash \mathcal{T}' /2$	11	24	46	80
$ G' \backslash M^{\nu_1} \times M^{\nu_2} \times M^{\nu_3} $ ($\nu_1 = \nu_2 = \nu_3$)	5	6	16	20
$ G' \backslash M^{\nu_1} \times M^{\nu_2} \times M^{\nu_3} $ ($\nu_i \neq \nu_j$ for some i, j)	2	6	10	20

Let $t = (U_0, U_d, V)$ with $d = n - a - b$ and $V = V(a, b, c_+, c_-)_{\text{even}}^0$. The following proposition is proved in the same way as Theorem 1.8.

Proposition 1.22. (i) *For every full flag \mathcal{F} in $M_0(V) \cap M'_0$, there exists a $g \in R(t) \cap G'$ such that $g\mathcal{F}$ is standard.*

(ii) *Let \mathcal{F} and \mathcal{F}' be two standard full flags in $M_0(V) \cap M'_0$. Then $\mathcal{F}' = g\mathcal{F}$ for some $g \in R(t) \cap G' \implies \mathcal{F}' = g_L \mathcal{F}$ for some $g_L \in L_V \cap G'$.*

(iii) *Let \mathcal{F} be a standard full flag in $M_0(V) \cap M'_0$. If $\mathbb{F} = \mathbb{F}_r$, then $|(R(t) \cap G')\mathcal{F}| = [r]_a [r]_b r^{\ell(\tau(\mathcal{F}))} |(L_V \cap G')\mathcal{F}|.$*

As in Section 1.4, we can express G' -orbits on $\mathcal{T}'_0 = M' \times M' \times M'_0$ by words with letters $\alpha, \beta, +, -, a, b, \dots$ and A, B, \dots . The following corollary is proved in the same way as Theorem 1.19.

Corollary 1.23. (i) $|\tilde{G}' \backslash \mathcal{T}'_0| = \frac{|G' \backslash \mathcal{T}'_0|}{2} = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^{n-2k} \binom{n}{2k} \frac{(2k)!}{k!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{4^{n-2k} n!}{k!(n-2k)!}.$

(ii) $|G' \backslash M^{\nu_1} \times M^{\nu_2} \times M_0^{\nu_3}| = \frac{|\tilde{G}' \backslash \mathcal{T}'_0|}{4} + \mu \frac{n!}{4(n/2)!}$ where

$$\mu = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even and } \nu_1 = \nu_2, \\ -1 & \text{if } n \text{ is even and } \nu_1 \neq \nu_2. \end{cases}$$

(iii) $|\mathrm{GL}_n(\mathbb{F}) \backslash M'_0| = 2|\mathrm{GL}_n(\mathbb{F}) \backslash M_0^0| = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{n-2k} n!}{k!(n-2k)!}.$

For $n = 2, 3, 4, 5$, the number of orbits is as follows.

n	2	3	4	5
$ \tilde{G}' \backslash \mathcal{T}'_0 = G' \backslash \mathcal{T}'_0 /2$	18	88	460	2544
$ G' \backslash M^{\nu_1} \times M^{\nu_2} \times M_0^{\nu_3} $ ($\nu_1 = \nu_2$)	5	22	118	636
$ G' \backslash M^{\nu_1} \times M^{\nu_2} \times M_0^{\nu_3} $ ($\nu_1 \neq \nu_2$)	4	22	112	636
$ \mathrm{GL}_n(\mathbb{F}) \backslash M'_0 = 2 \mathrm{GL}_n(\mathbb{F}) \backslash M_0^0 $	6	20	76	312

Remark 1.24. When $\mathbb{F} = \mathbb{C}$, $\mathrm{GL}_n(\mathbb{C})$ is a symmetric subgroup of $\mathrm{SO}_{2n}(\mathbb{C})$. The structure of $\mathrm{GL}_n(\mathbb{C}) \backslash M_0^0 \cong \mathrm{GL}_n(\mathbb{C}) \backslash \mathrm{SO}_{2n}(\mathbb{C})/B$ is described in [MO90] (Fig.19 and Fig.20) for $n = 3$ and 4. (In [MO90], read $GL(n, \mathbb{C})$ for $\mathbb{C}^\times \times PSL(n, \mathbb{C})$. For $A++$ in Fig.20 read $AA+$.)

2. ORBITS ON M AND M_0

2.1. Preliminaries. First we prepare some results on orbits on M and M_0 which follow essentially from the Bruhat decompositions for the Chevalley-type groups. Since we need more explicit results, we will prove them by elementary arguments.

Write $W_0 = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_{n-d}$, $W_1 = \mathbb{F}e_{n-d+1} \oplus \dots \oplus \mathbb{F}e_{n+d+1}$ and $W_2 = \mathbb{F}e_{n+d+2} \oplus \dots \oplus \mathbb{F}e_{2n+1}$. Then $W_0^\perp = W_0 \oplus W_1$. Define a maximal parabolic subgroup $P_{W_0} = \{g \in G \mid gW_0 = W_0\}$ of G . Let N_{W_0} , L_{W_0} and L_{W_1} be subgroups of P_{W_0} defined by

$$N_{W_0} = \left\{ \begin{pmatrix} I_m & * & * \\ 0 & I_{2d+1} & * \\ 0 & 0 & I_m \end{pmatrix} \right\}, \quad L_{W_0} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & I_{2d+1} & 0 \\ 0 & 0 & J_m {}^t A^{-1} J_m \end{pmatrix} \mid A \in \mathrm{GL}_m(\mathbb{F}) \right\}$$

and $L_{W_1} = \{g \in G \mid gv = v \text{ for all } v \in W_0 \oplus W_2\},$

respectively, where $m = n - d$. Then N_{W_0} is the unipotent radical of P_{W_0} and $L_{W_0}L_{W_1} \cong L_{W_0} \times L_{W_1}$ is a Levi subgroup of P_{W_0} . The subgroup L_{W_1} is identified with the special orthogonal group for W_1 . Define

$$Q = L_{W_0}N_{W_0} = N_{W_0}L_{W_0}.$$

Write

$$g(X, Z) = \begin{pmatrix} I_m & -J_m {}^t X J_{2d+1} & Z \\ 0 & I_{2d+1} & X \\ 0 & 0 & I_m \end{pmatrix}$$

where $X = \{x_{i,j}\}$ is a $(2d+1) \times m$ matrix and $Z = \{z_{i,j}\}$ is an $m \times m$ matrix. Then we can write

$$(2.1) \quad N_{W_0} = \{g(X, Z) \mid Z + J_m {}^t Z J_m = -J_m {}^t X J_{2d+1} X\}.$$

Noting that

$$(2.2) \quad z_{m+1-j,j} = -\frac{1}{2} \sum_{k=1}^{2d+1} x_{k,j} x_{2d+2-k,j}$$

for $j = 1, \dots, m$ and

$$(2.3) \quad z_{i,j} = -z_{m+1-j,m+1-i} - \sum_{k=1}^{2d+1} x_{k,j} x_{2d+2-k,m+1-i}$$

for $i + j \geq m + 2$, we have:

Lemma 2.1. *There exists a bijection between $\mathbb{F}^{m(2d+1)+m(m-1)/2}$ and N_{W_0} given by*

$$(X, \{z_{i,j}\}_{i+j \leq m}) \mapsto g(X, Z).$$

Definition 2.2. A maximally isotropic subspace V in \mathbb{F}^{2n+1} is called standard if

$$\begin{aligned} V &= (V \cap W_0) \oplus (V \cap W_1) \oplus (V \cap W_2), \\ V \cap W_0 &= U_{(\alpha)} (= \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_a) \\ \text{and } V \cap W_2 &= U_{(\beta)} (= \mathbb{F}e_{n+d+2} \oplus \dots \oplus \mathbb{F}e_{2n-a+1}) \end{aligned}$$

with some $a = 0, \dots, m = n - d$.

Proposition 2.3. (i) *For every $V \in M$, there exists a $g \in Q$ such that gV is standard.*

(ii) *Let V and V' be two standard elements in M . If $V' = gV$ for some $g = g_Q g_L \in P_{W_0} = Q L_{W_1}$, then $V' = g_L V$.*

(iii) *When $\mathbb{F} = \mathbb{F}_r$, $|QV| = r^{((n-a)(n-a+1)-d(d+1))/2} \frac{[r]_{a+b}}{[r]_a [r]_b}$ for $V \in M$. Here $a = \dim(V \cap W_0)$, $b = n - d - a$ and $[r]_k = (r+1)(r^2+r+1) \dots (r^{k-1} + r^{k-2} + \dots + 1)$.*

Proof. (i) Let $\pi_2 : \mathbb{F}^{2n+1} \rightarrow W_2$ denote the projection with respect to the direct sum decomposition $\mathbb{F}^{2n+1} = W_0 \oplus W_1 \oplus W_2$. By the action of $L_{W_0} \cong \text{GL}(W_0)$, we may assume $V \cap W_0 = U_{(\alpha)}$. It is equivalent to $\pi_2(V) = U_{(\beta)}$. Take vectors $u_1, \dots, u_b \in V$ such that $\pi_2(u_j) = e_{n+d+1+j}$ for $j = 1, \dots, b$. Then we can write

$$u_j = e_{n+d+1+j} + \sum_{i=1}^m \tilde{z}_{i,j} e_i + \sum_{i=1}^{2d+1} \tilde{x}_{i,j} e_{m+i}$$

with some $\tilde{z}_{i,j}, \tilde{x}_{i,j} \in \mathbb{F}$. It follows from the condition $(u_j, u_k) = 0$ for $j, k = 1, \dots, b$ that

$$\tilde{z}_{m+1-j,j} = -\frac{1}{2} \sum_{k=1}^{2d+1} \tilde{x}_{k,j} \tilde{x}_{2d+2-k,j}$$

for $j = 1, \dots, b$ and that

$$\tilde{z}_{i,j} = -\tilde{z}_{m+1-j,m+1-i} - \sum_{k=1}^{2d+1} \tilde{x}_{k,j} \tilde{x}_{2d+2-k,m+1-i}$$

for $(i, j) \in \{(i, j) \mid i + j \geq m + 2, j \leq b\}$. Hence we can take $X = \{x_{i,j}\}$ and $Z = \{z_{i,j}\}$ satisfying (2.2) and (2.3) so that $x_{i,j} = \tilde{x}_{i,j}$ and that $z_{i,j} = \tilde{z}_{i,j}$ for $j = 1, \dots, b$. Take $g = g(X, Z) \in N_{W_0}$. Then we have

$$g^{-1}V = (g^{-1}V \cap (W_0 \oplus W_1)) \oplus (g^{-1}V \cap W_2) \quad \text{and} \quad g^{-1}V \cap W_2 = U_{(\beta)}.$$

Since $g^{-1}V \cap (W_0 \oplus W_1) \perp g^{-1}V \cap W_2$, we have

$$g^{-1}V \cap (W_0 \oplus W_1) \subset U_{(\alpha)} \oplus W_1.$$

Hence

$$g^{-1}V \cap (W_0 \oplus W_1) = (g^{-1}V \cap W_0) \oplus (g^{-1}V \cap W_1)$$

with $g^{-1}V \cap W_0 = V \cap W_0 = U_{(\alpha)}$.

(ii) The condition $V' = gV$ implies $\dim(V \cap W_0) = \dim(V' \cap W_0)$. So we have

$$V_i \cap W_0 = V'_i \cap W_0 = U_{(\alpha)} \quad \text{and} \quad V_i \cap W_2 = V'_i \cap W_2 = U_{(\beta)}.$$

We have only to show that $g_L(V \cap W_1) = V' \cap W_1$. Let v be an element of $V \cap W_1$. Then we have

$$gv \in V' \cap W_0^\perp = (V' \cap W_0) \oplus (V' \cap W_1).$$

So we can write $gv = v_0 + v_1$ with some $v_0 \in V' \cap W_0$ and $v_1 \in V' \cap W_1$. Since $g_L v \in W_1$ and since $gv = g_Q g_L v \in g_L v + W_0$, it follows that $g_L v = v_1$. Thus we have proved $g_L(V \cap W_1) = V' \cap W_1$ since $\dim(V \cap W_1) = \dim(V' \cap W_1)$.

(iii) By (i), we may assume that V is standard. Note that we may consider ℓV instead of V with some $\ell \in L_{W_1}$ because $\ell Q \ell^{-1} = Q$. So we may assume

$$V \cap W_1 = \mathbb{F}e_{m+1} \oplus \dots \oplus \mathbb{F}e_n.$$

Let $g = g(X, Z)$ be an element of N_{W_0} such that $gV = V$. Since $g(V \cap W_2) \subset V$, we have

$$(2.4) \quad i \geq d + 1, j \leq b \implies x_{i,j} = 0$$

and

$$(2.5) \quad i \geq a + 1, j \leq b \implies z_{i,j} = 0.$$

It follows from (2.2), (2.3) and (2.4) that $z_{m+1-j,j} = 0$ for $j = 1, \dots, b$ and that $z_{i,j} = -z_{m+1-j,m+1-i}$ for $(i, j) \in \{(i, j) \mid j \leq b, i + j \geq m + 2\}$. Hence the condition (2.5) follows from the condition (2.4) and the condition

$$(2.6) \quad i \geq a + 1, j \leq b, i + j \leq a + b \implies z_{i,j} = 0.$$

Conversely suppose that $g = g(X, Z)$ satisfies the conditions (2.4) and (2.6). Then $g(V \cap W_2) \subset V$. Write

$$ge_{m+j} = e_{m+j} + \sum_{i=1}^m y_{i,j} e_i$$

for $j = 1, \dots, 2d + 1$. Then $y_{i,j} = -x_{2d+2-j,m+1-i}$ by (2.1). Since

$$y_{i,j} = -x_{2d+2-j,m+1-i} = 0 \text{ for } (i, j) \in \{(i, j) \mid i \geq a + 1, j \leq d + 1\}$$

by (2.4), we also have $g(V \cap W_1) \subset V$. Hence $gV = V$.

Thus it follows from Lemma 2.1 that

$$\begin{aligned} |N_{W_0} V| &= |N_{W_0}| / |\{g \in N_{W_0} \mid gV = V\}| = r^{b(d+1)+b(b-1)/2} \\ &= r^{((n-a)(n-a+1)-d(d+1))/2}. \end{aligned}$$

On the other hand, we have $|L_{W_0} V| = [r]_{a+b}/[r]_a[r]_b$ since $L_{W_0} \cong \text{GL}(W_0)$ and $h(V \cap W_2)$ is the orthogonal subspace of $h(V \cap W_0)$ in W_2 for $h \in L_0$. Thus we have the desired formula for $|QV|$. \square

Fix a standard maximally isotropic subspace V in \mathbb{F}^{2n+1} and let $M_0(V)$ denote the subvariety of M_0 consisting of full flags $\mathcal{F} : V_1 \subset \dots \subset V_n$ such that $V_n = V$.

For a full flag $\mathcal{F} : V_1 \subset \dots \subset V_n$ contained in $M_0(V)$, define

$$a_i = a_i(\mathcal{F}) = \dim(V_i \cap W_0), \quad c_i = c_i(\mathcal{F}) = \dim(V_i \cap W_0^\perp) - \dim(V_i \cap W_0)$$

$$\text{and } b_i = b_i(\mathcal{F}) = \dim V_i - \dim(V_i \cap W_0^\perp)$$

for $i = 0, 1, \dots, n$. Define subsets

$$\begin{aligned} I_{(\alpha)} &= \{\alpha_1, \dots, \alpha_a\} = \{i \in I \mid a_i = a_{i-1} + 1\}, \\ I_{(\lambda)} &= \{\lambda_1, \dots, \lambda_d\} = \{i \in I \mid c_i = c_{i-1} + 1\}, \\ I_{(\beta)} &= \{\beta_1, \dots, \beta_b\} = \{i \in I \mid b_i = b_{i-1} + 1\} \end{aligned}$$

of $I = \{1, \dots, n\}$ with $\alpha_1 < \dots < \alpha_a$, $\lambda_1 < \dots < \lambda_d$ and $\beta_1 < \dots < \beta_b$. Then $I = I_{(\alpha)} \sqcup I_{(\lambda)} \sqcup I_{(\beta)}$. Consider the permutation

$$\sigma = \sigma(\mathcal{F}) : (1 \, 2 \, \dots \, n) \mapsto (\alpha_1 \dots \alpha_a \lambda_1 \dots \lambda_d \beta_1 \dots \beta_b)$$

of I . Then the inversion number $\ell(\sigma)$ of σ is

$$\ell(\sigma) = |\{(i, j) \mid \alpha_i > \lambda_j\}| + |\{(i, j) \mid \alpha_i > \beta_j\}| + |\{(i, j) \mid \lambda_i > \beta_j\}|.$$

Definition 2.4. A full flag $\mathcal{F} : V_1 \subset \cdots \subset V_n = V$ in $M_0(V)$ is called L_{W_1} -standard if

$$\begin{aligned} V_i &= (V_i \cap W_0) \oplus (V_i \cap W_1) \oplus (V_i \cap W_2), \\ V_i \cap W_0 &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_i(\mathcal{F})} \\ \text{and } V_i \cap W_2 &= \mathbb{F}e_{n+d+2} \oplus \cdots \oplus \mathbb{F}e_{n+d+1+b_i(\mathcal{F})} \end{aligned}$$

for $i = 1, \dots, n$.

Write $Q_V = Q \cap P_V$ and $(L_{W_1})_V = L_{W_1} \cap P_V$. By Proposition 2.3 (ii), we have

$$P_{W_0} \cap P_V = Q_V(L_{W_1})_V.$$

Proposition 2.5. (i) For every full flag $\mathcal{F} : V_1 \subset \cdots \subset V_n = V$ in $M_0(V)$, there exists a $g \in Q_V$ such that $g\mathcal{F} : gV_1 \subset \cdots \subset gV_n = V$ is L_{W_1} -standard.

(ii) Let \mathcal{F} and \mathcal{F}' be two L_{W_1} -standard full flags in $M_0(V)$. If $\mathcal{F}' = g\mathcal{F}$ for some $g = g_Q g_L \in P_{W_0} \cap P_V = Q_V(L_{W_1})_V$, then $\mathcal{F}' = g_L \mathcal{F}$.

(iii) When $\mathbb{F} = \mathbb{F}_r$, we have $|Q_V \mathcal{F}| = r^{\ell(\sigma(\mathcal{F}))} [r]_a [r]_b$.

Proof. (i) Since V is standard, it is written as

$$V = (V \cap W_0) \oplus (V \cap W_1) \oplus (V \cap W_2),$$

with $V \cap W_0 = U_{(\alpha)}$ and $V \cap W_2 = U_{(\beta)}$. We may moreover assume $V \cap W_1 = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_n$ replacing V by ℓV with some $\ell \in L_{W_1}$. By the same reason, we may assume

$$\pi_1(V_{\lambda_i} \cap (W_0 \oplus W_1)) = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_{m+i}$$

for $i = 1, \dots, d$ where $\pi_1 : W_0 \oplus W_1 \rightarrow W_1$ is the projection.

By the action of $L_{W_0} \cap P_V$, we may assume

$$V_i \cap W_0 = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_i(\mathcal{F})} \quad \text{and} \quad \pi_2(V_i) = \mathbb{F}e_{n+d+2} \oplus \cdots \oplus \mathbb{F}e_{n+d+1+b_i(\mathcal{F})}$$

for $i = 1, \dots, n$. We can take vectors

$$w_j = e_{m+j} + \sum_{i=1}^a \tilde{y}_{i,j} e_i \in (V \cap W_0) \oplus (V \cap W_1)$$

for $j = 1, \dots, d$ such that $\mathbb{F}w_1 \oplus \cdots \oplus \mathbb{F}w_j \subset V_{\lambda_j}$. We can also take vectors

$$w'_j = e_{n+d+1+j} + \sum_{i=1}^a \tilde{z}_{i,j} e_i + \sum_{i=1}^d \tilde{x}_{i,j} e_{m+i} \in V$$

for $j = 1, \dots, b$ such that $\mathbb{F}w'_1 \oplus \cdots \oplus \mathbb{F}w'_j \subset V_{\beta_j}$. Take $g = g(X, Z) \in N_{W_0} \cap P_V$ with $X = \{x_{i,j}\}$ and $Z = \{z_{i,j}\}$ so that

$$\begin{aligned} x_{i,j} &= \begin{cases} \tilde{x}_{i,j} & \text{for } i \leq d, j \leq b, \\ 0 & \text{for } i \geq d+1, j \leq b, \\ 0 & \text{for } i \leq d+1, j \geq b+1, \\ \tilde{y}_{m+1-j, 2d+2-i} & \text{for } i \geq d+2, j \geq b+1, \end{cases} \\ z_{i,j} &= \begin{cases} \tilde{z}_{i,j} & \text{for } i \leq a, j \leq b, \\ 0 & \text{if } i \geq a+1 \text{ or } j \geq b+1. \end{cases} \end{aligned}$$

Then we have

$$g^{-1}V_i = (g^{-1}V_i \cap W_0) \oplus (g^{-1}V_i \cap W_1) \oplus (g^{-1}V_i \cap W_2)$$

with $g^{-1}V_i \cap W_0 = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_i(\mathcal{F})}$, $g^{-1}V_i \cap W_1 = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_{m+c_i(\mathcal{F})}$ and $g^{-1}V_i \cap W_2 = \mathbb{F}e_{n+d+2} \oplus \cdots \oplus \mathbb{F}e_{n+d+1+b_i(\mathcal{F})}$ for $i = 1, \dots, n$.

(ii) The condition $\mathcal{F}' = g\mathcal{F}$ implies $a_i(\mathcal{F}) = a_i(\mathcal{F}')$ and $b_i(\mathcal{F}) = b_i(\mathcal{F}')$ for $i = 1, \dots, n$. So we have

$$V_i \cap W_0 = V'_i \cap W_0 \quad \text{and} \quad V_i \cap W_2 = V'_i \cap W_2.$$

We have only to show that $g_L(V_i \cap W_1) = V'_i \cap W_1$ for $i = 1, \dots, n$. Let v be an element of $V_i \cap W_1$. Then we have

$$gv \in V'_i \cap W_0^\perp = (V'_i \cap W_0) \oplus (V'_i \cap W_1).$$

So we can write $gv = v_0 + v_1$ with some $v_0 \in V'_i \cap W_0$ and $v_1 \in V'_i \cap W_1$. Since $g_L v \in W_1$ and since $gv = g_Q g_L v \in g_L v + W_0$, it follows that $g_L v = v_1$. Thus we have proved $g_L(V_i \cap W_1) = V'_i \cap W_1$ since $\dim(V_i \cap W_1) = \dim(V'_i \cap W_1)$.

(iii) By (i), we may assume that \mathcal{F} is L_{W_1} -standard. As in the proof of (i), we may assume $V_{\lambda_i} \cap W_1 = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_{m+i}$ for $i = 1, \dots, d$. Note that

$$gV = V \iff (2.4) \text{ and } (2.5)$$

for $g = g(X, Z) \in N_{W_0}$.

Suppose $g\mathcal{F} = \mathcal{F}$. Then since $ge_{m+j} \in V_{\lambda_j}$, we have

$$i \leq a, \quad \alpha_i > \lambda_j \implies y_{i,j} = -x_{2d+2-j, m+1-i} = 0$$

for $j = 1, \dots, d$. Since $ge_{n+d+1+j} \in V_{\beta_j}$, we also have

$$i \leq d, \quad \lambda_i > \beta_j \implies x_{i,j} = 0$$

$$\text{and} \quad i \leq a, \quad \alpha_i > \beta_j \implies z_{i,j} = 0$$

for $j = 1, \dots, b$. Conversely if $g \in N_{W_0} \cap P_V$ satisfies the above three conditions, then $g\mathcal{F} = \mathcal{F}$.

Thus we have

$$|(N_{W_0} \cap P_V)\mathcal{F}| = |N_{W_0} \cap P_V| / |\{g \in N_{W_0} \cap P_V \mid g\mathcal{F} = \mathcal{F}\}| = r^{\ell(\sigma(\mathcal{F}))}.$$

Clearly $|(L_{W_0} \cap P_V)\mathcal{F}| = [r]_a[r]_b$ (the product of the numbers of full flags in $V \cap W_0$ and $V \cap W_2$). So we have $|Q_V\mathcal{F}| = r^{\ell(\sigma(\mathcal{F}))}[r]_a[r]_b$. \square

2.2. First reduction. First apply Proposition 2.3 to the case of $d = 0$. Noting that $Q = P_{W_0} = P$ in this case, we get the P -orbit decomposition

$$(2.7) \quad M = \bigsqcup_{i=0}^n PU_i$$

of M . (Of course, we can also deduce this from the Bruhat decomposition of G .) Furthermore if $\mathbb{F} = \mathbb{F}_r$, then

$$(2.8) \quad |PU_i| = r^{i(i+1)/2} \frac{[r]_n}{[r]_i[r]_{n-i}}$$

by Proposition 2.3 (iii). By (2.7), we have a decomposition $M \times M = \bigsqcup_{i=0}^n \{(gU_0, gU_i) \mid g \in G\}$. Since the isotropy subgroup of G at $(U_0, U_i) \in M \times M$ is $R_i = P \cap P_{U_i}$, we can write

$$(2.9) \quad M \times M \cong \bigsqcup_{i=0}^n G/R_i.$$

By (2.9), we have only to describe G -orbits on $(G/R_i) \times M$ and $(G/R_i) \times M_0$ with respect to the diagonal action of G for $i = 0, \dots, n$. These orbit decompositions are identified with the R_i -orbit decompositions of M and M_0 , respectively.

2.3. Second reduction. Consider R_d -orbit decompositions of M and M_0 for $d = 0, \dots, n$. Since $U_0 \cap U_d = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_{n-d} = W_0$, we have $Q \subset R_d \subset P_{W_0}$ in the setting of Section 2.1. So we can apply the results in Section 2.1 once more. By Proposition 2.3 and Proposition 2.5, the problems are reduced to the $L_{W_1} \cap R_d$ -orbit decompositions of M and M_0 . Note that

$$L_{W_1} \cap R_d = \{g \in L_{W_1} \mid g(W_1 \cap U_0) = W_1 \cap U_0 \text{ and } g(W_1 \cap U_d) = W_1 \cap U_d\} \cong \mathrm{GL}_d(\mathbb{F})$$

since $(W_1 \cap U_0) \cap (W_1 \cap U_d) = \{0\}$. So we will consider the case of $d = n$ in the next subsection.

2.4. The case of $d = n$. Assume $d = n$. For $A \in \mathrm{GL}_n(\mathbb{F})$, define

$$h[A] = \begin{pmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & J^t A^{-1} J \end{pmatrix} \text{ with } J = J_n = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}$$

as in Section 1.1. Then we can write $R_n = \{h[A] \mid A \in \mathrm{GL}_n(\mathbb{F})\} \cong \mathrm{GL}_n(\mathbb{F})$. Write $H = R_n$ in this subsection. For a maximal isotropic subspace V in \mathbb{F}^{2n+1} , define

$$c_+ = c_+(V) = \dim(V \cap U_0), \quad c_- = c_-(V) = \dim(V \cap U_n), \quad c_0 = c_0(V) = n - c_+ - c_-$$

$$\text{and } \varepsilon(V) = \dim(V + (U_0 \oplus U_n)) - 2n \in \{0, 1\}.$$

Let $M(U_{(+)}, U_{(-)})$ denote the subvariety of M consisting of $V \in M$ such that

$$V \cap U_0 = U_{(+)} = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_{c_+} \text{ and that } V \cap U_n = U_{(-)} = \mathbb{F}e_{n+2} \oplus \dots \oplus \mathbb{F}e_{n+c_-+1}.$$

By the action of $H \cong \mathrm{GL}_n(\mathbb{F})$, we may assume $V \in M(U_{(+)}, U_{(-)})$. Let $\pi_+ : \mathbb{F}^{2n+1} \rightarrow U_0$, $\pi_- : \mathbb{F}^{2n+1} \rightarrow U_n$ and $\pi_Z : \mathbb{F}^{2n+1} \rightarrow Z = \mathbb{F}e_{n+1}$ denote the projections with respect to the direct sum decomposition $\mathbb{F}^{2n+1} = U_0 \oplus U_n \oplus Z$. Write $U_{(0+)} = \mathbb{F}e_{c_++1} \oplus \dots \oplus \mathbb{F}e_{c_++c_0}$ and $U_{(0-)} = \mathbb{F}e_{n+c_-+2} \oplus \dots \oplus \mathbb{F}e_{n+c_-+c_0+1}$. Then

$$U_{(0)} = U_{(0+)} \oplus U_{(0-)} \oplus Z.$$

Lemma 2.6. (i) $\ker \pi_+|_V = U_{(-)}$ and $\ker \pi_-|_V = U_{(+)}$.

(ii) $\pi_+(V) = U_{(+)} \oplus U_{(0+)}$ and $\pi_-(V) = U_{(-)} \oplus U_{(0-)}.$

(iii) The inner product $(\ , \)$ is nondegenerate on the pair $(U_{(0+)}, U_{(0-)})$.

Proof. (i) By the symmetry, we have only to prove the first equality. The kernel of the map $\pi_+|_V : V \rightarrow U_0$ is $V \cap (U_n \oplus Z)$. If an element $v \in V$ is written as $v = v_- + z$ with $v_- \in U_n$ and $z \in Z$, then we have

$$0 = (v, v) = (v_- + z, v_- + z) = (z, z)$$

and hence $z = 0$. Thus we have $V \cap (U_n \oplus Z) = V \cap U_n = U_{(-)}$.

(ii) For $u \in V$ and $v \in U_{(-)}$, we have

$$0 = (u, v) = (\pi_+(u) + \pi_-(u) + \pi_Z(u), v) = (\pi_+(u), v).$$

Hence $\pi_+(V) \subset U_{(+)} \oplus U_{(0+)}$. On the other hand, the dimension of $\pi_+(V)$ is $n - \dim U_{(-)} = n - c_-$ by (i). Hence the equality holds. By the symmetry, we have the second formula.

(iii) is clear from the definition. \square

Corollary 2.7. $V = U_{(+)} \oplus U_{(-)} \oplus (V \cap U_{(0)})$.

By Lemma 2.6 (i) and (ii), we have linear isomorphisms

$$\pi_+|_{V \cap U_{(0)}} : V \cap U_{(0)} \rightarrow U_{(0+)} \quad \text{and} \quad \pi_-|_{V \cap U_{(0)}} : V \cap U_{(0)} \rightarrow U_{(0-)}.$$

So we can define a linear isomorphism

$$f_V = \pi_-|_{V \cap U_{(0)}} \circ (\pi_+|_{V \cap U_{(0)}})^{-1} : U_{(0+)} \rightarrow U_{(0-)}.$$

We also define a linear form $\varphi_V : U_{(0+)} \rightarrow \mathbb{F}$ by

$$\varphi_V(v) = (e_{n+1}, \pi_+|_{V \cap U_{(0)}}^{-1}(v)).$$

By Lemma 2.6 (iii) and the above argument, we can define a nondegenerate bilinear form on $U_{(0+)}$ by

$$\langle u, v \rangle_V = (f_V(u), v)$$

for $u, v \in U_{(0+)}$. Define the alternating part and the symmetric part of $\langle \cdot, \cdot \rangle_V$ by

$$\langle u, v \rangle_V^{\text{alt}} = \frac{1}{2}(\langle u, v \rangle_V - \langle v, u \rangle_V) \quad \text{and} \quad \langle u, v \rangle_V^{\text{sym}} = \frac{1}{2}(\langle u, v \rangle_V + \langle v, u \rangle_V),$$

respectively. Let $u = u_+ + u_Z + u_-$ and $v = v_+ + v_Z + v_-$ be elements of $V \cap U_{(0)}$ with $u_+, v_+ \in U_{(0+)}$, $u_Z, v_Z \in Z$ and $u_-, v_- \in U_{(0-)}$. Then we have

$$\begin{aligned} 0 &= (u, v) = (u_+ + u_Z + u_-, v_+ + v_Z + v_-) \\ &= (u_+, v_-) + (u_Z, v_Z) + (u_-, v_+) \\ &= (u_+, f_V(v_+)) + (u_Z, v_Z) + (f_V(u_+), v_+). \end{aligned}$$

Hence we have

$$(2.10) \quad \langle u, v \rangle_V^{\text{sym}} = \frac{1}{2}(\langle u, v \rangle_V + \langle v, u \rangle_V) = -\frac{1}{2}\varphi_V(u)\varphi_V(v)$$

for $u, v \in U_{(0+)}$. In particular, if u or v is in the kernel of φ_V , then

$$(2.11) \quad \langle u, v \rangle_V = \langle u, v \rangle_V^{\text{alt}}.$$

Lemma 2.8. (i) If $c_0(V) = 0$, then $\varepsilon(V) = 0$.

(ii) If $c_0(V)$ is odd, then $\varepsilon(V) = 1$.

Proof. (i) If $c_0(V) = 0$, then $V = U_{(+)} \oplus U_{(-)}$. Hence $V \subset U_0 \oplus U_n$.

(ii) If $\varepsilon(V) = 0$, then the bilinear form $\langle \cdot, \cdot \rangle_V$ is alternating by (2.11). Since it is also nondegenerate, the dimension $c_0(V) = \dim U_{(0+)}$ is even. \square

Proposition 2.9. (i) *If c_0 is even, then $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ is nondegenerate on $U_{(0+)}$.*

(ii) *If c_0 is odd, then $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ is nondegenerate on $\varphi_V^{-1}(0)$.*

Proof. (i) By (2.11), we may assume that φ_V is nontrivial. Suppose that $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ is degenerate. Then the subspace

$$Y = \{y \in U_{(0+)} \mid \langle y, v \rangle_V^{\text{alt}} = 0 \text{ for all } v \in U_{(0+)}\}$$

of $U_{(0+)}$ is nontrivial and even-dimensional. Take a nonzero element y of $Y \cap \varphi_V^{-1}(0)$. Then for all $v \in U_{(0+)}$, it follows from (2.11) that $\langle y, v \rangle_V = \langle y, v \rangle_V^{\text{alt}} = 0$, contradicting that $\langle \cdot, \cdot \rangle_V$ is nondegenerate on $U_{(0+)}$.

(ii) By (2.11), $\langle \cdot, \cdot \rangle_V = \langle \cdot, \cdot \rangle_V^{\text{alt}}$ on $\varphi_V^{-1}(0)$. Suppose that it is degenerate on $\varphi_V^{-1}(0)$. Then the subspace

$$Y = \{y \in \varphi_V^{-1}(0) \mid \langle y, v \rangle_V = 0 \text{ for all } v \in \varphi_V^{-1}(0)\}$$

of $\varphi_V^{-1}(0)$ is nontrivial and even-dimensional. Take a $v_0 \in U_{(0+)} - \varphi_V^{-1}(0)$ and write

$$Y' = \{y \in Y \mid \langle y, v_0 \rangle_V = 0\}.$$

Then Y' is nontrivial and $\langle y, v \rangle_V = 0$ for all $v \in U_{(0+)}$, contradicting that $\langle \cdot, \cdot \rangle_V$ is nondegenerate on $U_{(0+)}$. \square

Proposition 2.10. *Let V and V' be two elements of $M(U_{(+)}, U_{(-)})$. Then the following three conditions are equivalent:*

(i) $V = V'$.

(ii) $f_V = f_{V'}$ and $\varphi_V = \varphi_{V'}$.

(iii) $\langle \cdot, \cdot \rangle_V^{\text{alt}} = \langle \cdot, \cdot \rangle_{V'}^{\text{alt}}$ and $\varphi_V = \varphi_{V'}$.

Proof. (ii) \implies (i). By the direct sum decomposition $U_{(0)} = U_{(0+)} \oplus Z \oplus U_{(0-)}$, $V \cap U_{(0)}$ is written as

$$V \cap U_{(0)} = \{(\pi_+(v), \pi_Z(v), \pi_-(v)) \mid v \in V \cap U_{(0)}\} = \{(u, \varphi_V(u)e_n, f_V(u)) \mid u \in U_{(0+)}\}.$$

Hence $V \cap U_{(0)}$ is determined by the two maps f_V and φ_V .

(iii) \implies (ii). By (2.10), $\langle \cdot, \cdot \rangle_V^{\text{sym}}$ is determined by φ_V . Hence the bilinear form $\langle \cdot, \cdot \rangle_V$ is determined by $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ and φ_V . Since the inner product (\cdot, \cdot) defines a nondegenerate pairing between $U_{(0+)}$ and $U_{(0-)}$, the map f_V is determined by $\langle \cdot, \cdot \rangle_V$.

The implication (i) \implies (iii) is trivial. \square

Proposition 2.11. (i) *Let V be an element of M . Write $c_{\pm} = c_{\pm}(V)$, $c_0 = c_0(V)$ and $\varepsilon = \varepsilon(V)$. Then*

$$V \in \begin{cases} HV(0, 0, c_+, c_-)_{\text{odd}} & \text{if } c_0 \text{ is odd,} \\ HV(0, 0, c_+, c_-)_{\text{even}}^0 & \text{if } c_0 \text{ is even and } \varepsilon = 0, \\ HV(0, 0, c_+, c_-)_{\text{even}}^1 & \text{if } c_0 \text{ is even and } \varepsilon = 1. \end{cases}$$

(ii) Let $L_0 \cong \mathrm{GL}_{c_0}(\mathbb{F})$ be as in Section 1.2. Then

$$\{\ell \in L_0 \mid \ell V = V\} \cong \begin{cases} 1 \times \mathrm{Sp}_{c_0-1}(\mathbb{F}) & \text{if } V = V(0, 0, c_+, c_-)_{\text{odd}}, \\ \mathrm{Sp}_{c_0}(\mathbb{F}) & \text{if } V = V(0, 0, c_+, c_-)_{\text{even}}^0, \\ Q_{c_0} & \text{if } V = V(0, 0, c_+, c_-)_{\text{even}}^1 \end{cases}$$

where $Q_{c_0} = \{g \in \mathrm{Sp}_{c_0}(\mathbb{F}) \mid gv = v\}$ with some nonzero $v \in \mathbb{F}^{c_0}$.

(iii) If $\mathbb{F} = \mathbb{F}_r$, then

$$|HV| = \frac{[r]_n}{[r]_{c_+}[r]_{c_-}[r]_{c_0}} \psi_{c_0}^\varepsilon(r).$$

Proof. (i) By the action of $H = R_n$, we may assume $V \cap U_0 = U_{(+)}$ and $V \cap U_n = U_{(-)}$. By the above arguments, the space V defines an alternating form $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ on $U_{(0+)}$ and a linear form $\varphi_V : U_{(0+)} \rightarrow \mathbb{F}$.

For $V = V(0, 0, c_+, c_-)_{\text{even}}^\varepsilon$, the form $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ is standard:

$$\langle e_{c_++i}, e_{c_++j} \rangle_V^{\text{alt}} = \langle e_{c_++i}, e_{c_++j} \rangle_0 = \begin{cases} \delta_{i, c_0+1-j} & \text{if } i \leq c_0/2, \\ -\delta_{i, c_0+1-j} & \text{if } i > c_0/2. \end{cases}$$

The linear form φ_V vanishes for $V = V(0, 0, c_+, c_-)_{\text{even}}^0$ and $\varphi_V(e_{c_++i}) = \delta_{i, c_0}$ for $V = V(0, 0, c_+, c_-)_{\text{even}}^1$. On the other hand, for $V = V(0, 0, c_+, c_-)_{\text{odd}}$ ($c_0 = 2c_1 - 1$), we have

$$\langle e_{c_++i}, e_{c_++j} \rangle_V^{\text{alt}} = \langle e_{c_++i}, e_{c_++j} \rangle'_0 = \begin{cases} \delta_{i, c_0-j} & \text{if } i < c_1, \\ 0 & \text{if } i = c_1, \\ -\delta_{i, c_0-j} & \text{if } i > c_1, \end{cases}$$

and $\varphi_V(e_{c_++i}) = \delta_{i, c_1}$. Note that $\langle \cdot, \cdot \rangle'_0$ is nondegenerate on $W = \varphi_V^{-1}(0) = \mathbb{F}e_{c_++1} \oplus \cdots \oplus \mathbb{F}e_{c_++c_1-1} \oplus \mathbb{F}e_{c_++c_1+1} \oplus \cdots \oplus \mathbb{F}e_{c_++c_0}$.

Let L_0 be the subgroup of H defined by

$$L_0 = \left\{ h \left[\begin{pmatrix} I_{c_+} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{c_-} \end{pmatrix} \right] \mid A \in \mathrm{GL}_{c_0}(\mathbb{F}) \right\} \cong \mathrm{GL}(U_{(0+)})$$

as in Section 1.2. By Proposition 2.10, we have only to take an $\ell \in L_0$ such that $\langle \cdot, \cdot \rangle_{\ell V}^{\text{alt}}$ and $\varphi_{\ell V}$ are standard. Here we note that

$$\begin{aligned} \langle u, v \rangle_{\ell V}^{\text{alt}} &= \frac{1}{2}((f_{\ell V}(u), v) - (u, f_{\ell V}(v))) \\ &= \frac{1}{2}(((\ell \circ f_V \circ \ell^{-1})(u), v) - (u, (\ell \circ f_V \circ \ell^{-1})(v))) \\ &= \frac{1}{2}(((f_V \circ \ell^{-1})(u), \ell^{-1}(v)) - (\ell^{-1}(u), (f_V \circ \ell^{-1})(v))) \\ &= \langle \ell^{-1}u, \ell^{-1}v \rangle_V^{\text{alt}}. \end{aligned}$$

First suppose that c_0 is even. Then $\langle \cdot, \cdot \rangle_V^{\text{alt}}$ is nondegenerate on $U_{(0+)}$ by Proposition 2.9. So there exists an $\ell \in L_0$ such that $\langle \cdot, \cdot \rangle_{\ell V}^{\text{alt}}$ is equal to the standard alternating form $\langle \cdot, \cdot \rangle_0$ on $U_{(0+)}$. If $\varphi_{\ell V} = 0$, then we have proved that

$\ell V = V(0, 0, c_+, c_-)_{\text{even}}^0$. Suppose $\varphi_{\ell V} \neq 0$. Then we can take a nonzero element v_0 of $U_{(0+)}$ such that $\varphi_{\ell V}(v) = \langle v_0, v \rangle_0$ for all $v \in U_{(0+)}$. Let S denote the subgroup of L_0 defined by

$$(2.12) \quad S = \{g \in L_0 \mid \langle gu, gv \rangle_0 = \langle u, v \rangle_0 \text{ for all } u, v \in U_{(0+)}\} \cong \text{Sp}_{c_0}(\mathbb{F}).$$

Then we can take an element ℓ_0 of S such that $\ell_0 v_0 = e_{c_++1}$. We have

$$\varphi_{\ell_0 \ell V}(v) = \varphi_{\ell V}(\ell_0^{-1} v) = \langle v_0, \ell_0^{-1} v \rangle_0 = \langle \ell_0 v_0, v \rangle_0 = \langle e_{c_++1}, v \rangle_0.$$

Since $\langle e_{c_++1}, e_{c_++i} \rangle_0 = \delta_{i, c_0}$, we have proved $\ell_0 \ell V = V(0, 0, c_+, c_-)_{\text{even}}^1$.

Next suppose that c_0 is odd. Since the bilinear form $\langle \cdot, \cdot \rangle_V$ is nondegenerate on $U_{(0+)}$, we can take a nonzero element $v_0 \in U_{(0+)}$ such that $\varphi_V(v) = \langle v_0, v \rangle_V$ for all $v \in U_{(0+)}$. Since $\langle v_0, v \rangle_V = 0$ for all $v \in \varphi_V^{-1}(0)$, it follows that $v_0 \notin \varphi_V^{-1}(0)$. Take an element ℓ of L_0 such that

$$\ell v_0 = e_{c_++c_1} \text{ and that } \ell \varphi_V^{-1}(0) = W.$$

Then we have $\langle e_{c_++c_1}, v \rangle_{\ell V}^{\text{alt}} = 0$ for all $v \in U_{(0+)}$ and $\varphi_{\ell V}(e_{c_++i}) = \delta_{i, c_1}$. Since $\langle \cdot, \cdot \rangle_{\ell V}^{\text{alt}}$ is nondegenerate on $W = \varphi_{\ell V}^{-1}(0)$, we can take an element ℓ_0 of

$$L'_0 = \{\ell \in L_0 \mid \ell e_{c_++c_1} = e_{c_++c_1} \text{ and } \ell W = W\} \cong \text{GL}_{c_0-1}(\mathbb{F})$$

such that $\langle \cdot, \cdot \rangle_{\ell_0 \ell V}^{\text{alt}} = \langle \cdot, \cdot \rangle'_0$ on W . Thus we have $\ell_0 \ell V = V(0, 0, c_+, c_-)_{\text{odd}}$.

(ii) By Proposition 2.10, we have

$$\ell V = V \iff \langle \cdot, \cdot \rangle_{\ell V}^{\text{alt}} = \langle \cdot, \cdot \rangle_V^{\text{alt}} \text{ and } \varphi_{\ell V} = \varphi_V$$

for $\ell \in L_0$. If $V = V(0, 0, c_+, c_-)_{\text{even}}^0$, then $\langle \cdot, \cdot \rangle_V^{\text{alt}} = \langle \cdot, \cdot \rangle_0$ and $\varphi_V = 0$. Hence

$$\ell V = V \iff \ell \in S \cong \text{Sp}_{c_0}(\mathbb{F})$$

with the subgroup S of L_0 defined in (2.12). Next suppose that $V = V(0, 0, c_+, c_-)_{\text{even}}^1$. Then $\langle \cdot, \cdot \rangle_V^{\text{alt}} = \langle \cdot, \cdot \rangle_0$ and $\varphi_V(v) = \langle e_{c_++1}, v \rangle_0$ for $v \in U_{(0+)}$. Hence

$$\ell V = V \iff \ell \in S \text{ and } \ell e_{c_++1} = e_{c_++1}.$$

Finally suppose that $V = V(0, 0, c_+, c_-)_{\text{odd}}$. Then $\langle \cdot, \cdot \rangle_V^{\text{alt}} = \langle \cdot, \cdot \rangle'_0$ and $\varphi_V(e_{c_++i}) = \delta_{i, c_1}$ for $i = 1, \dots, c_0$. Hence

$$\ell V = V \iff \langle \ell u, \ell v \rangle'_0 = \langle u, v \rangle'_0 \text{ for } u, v \in U_{(0+)} \text{ and } \varphi_{\ell V} = \varphi_V \iff \ell \in S'$$

where $S' = \{\ell \in L_0 \mid \ell(e_{c_++c_0}) = e_{c_++c_0}, \ell(W) = W \text{ and } \langle \ell u, \ell v \rangle'_0 = \langle u, v \rangle'_0 \text{ for } u, v \in W\} \cong 1 \times \text{Sp}_{c_0-1}(\mathbb{F})$.

(iii) If we fix c_+ and c_- , then we have $[r]_n/[r]_{c_+}[r]_{c_-}[r]_{c_0}$ choices of the pair $(V \cap U_0, V \cap U_n)$. On the other hand, every element of $M(U_{(+)}, U_{(-)})$ is contained in the L_0 -orbit of $V = V(0, 0, c_+, c_-)_{\text{odd}}$, $V(0, 0, c_+, c_-)_{\text{even}}^0$ or $V(0, 0, c_+, c_-)_{\text{even}}^1$ as is proved in (i). Since $|L_0 V| = \psi_{c_0}^\varepsilon(r)$ by (ii), we have the desired formula for HV . \square

Let P_H be the parabolic subgroup of $H = R_n$ defined by

$$\begin{aligned} P_H &= \{g \in H \mid gU_{(+)} = U_{(+)} \text{ and } g(U_{(+)} \oplus U_{(0+)}) = U_{(+)} \oplus U_{(0+)}\} \\ &= \{g \in H \mid gU_{(+)} = U_{(+)} \text{ and } gU_{(-)} = U_{(-)}\}. \end{aligned}$$

Then the unipotent radical N_H of P_H is written as

$$N_H = \left\{ h \left[\begin{pmatrix} I_{c_+} & * & * \\ 0 & I_{c_0} & * \\ 0 & 0 & I_{c_-} \end{pmatrix} \right] \right\}$$

and

$$L = \left\{ h \left[\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \right] \mid A \in \mathrm{GL}_{c_+}(\mathbb{F}), B \in \mathrm{GL}_{c_0}(\mathbb{F}), C \in \mathrm{GL}_{c_-}(\mathbb{F}) \right\}$$

is a Levi subgroup of P_H . Let $\mathcal{M}(p, q; \mathbb{F})$ denote the space of $p \times q$ matrices with entries in \mathbb{F} . Then every element of N_H is written as

$$n(A, B, C) = h \left[\begin{pmatrix} I_{c_+} & A & C \\ 0 & I_{c_0} & B \\ 0 & 0 & I_{c_-} \end{pmatrix} \right]$$

with $A \in \mathcal{M}(c_+, c_0; \mathbb{F})$, $B \in \mathcal{M}(c_0, c_-; \mathbb{F})$ and $C \in \mathcal{M}(c_+, c_-; \mathbb{F})$.

Fix a $V \in M(U_{(+)}, U_{(-)})$. By Definition 1.6, a full flag $\mathcal{F} : V_1 \subset \cdots \subset V_n$ is called standard if

$$V_n = V \quad \text{and} \quad V_i = (V_i \cap U_{(+-)}) \oplus (V_i \cap U_{(0)})$$

for all $i = 1, \dots, n$. Here we write $U_{(+-)} = U_{(+)} \oplus U_{(-)}$.

Lemma 2.12. (i) For every $g \in N_H$, there exists a linear map

$$f_g : V \cap U_{(0)} \rightarrow U_{(+-)}$$

such that $gv = v + f_g(v)$ for all $v \in V \cap U_{(0)}$.

(ii) The map $g \mapsto f_g$ is a surjection of N_H onto the space of \mathbb{F} -linear maps $\mathrm{Hom}(V \cap U_{(0)}, U_{(+-)})$.

Proof. (i) For $g = n(A, B, C)$, we have $g\pi_+(v) = \pi_+(v) + A\pi_+(v)$ for $v \in V \cap U_{(0)}$. Here the matrix A is naturally identified with the linear map $A : U_{(0+)} \rightarrow U_{(+)}$. Since

$$J \begin{pmatrix} I_{c_+} & 0 & 0 \\ {}^t A & I_{c_0} & 0 \\ {}^t C & {}^t B & I_{c_-} \end{pmatrix}^{-1} J = \begin{pmatrix} I_{c_-} & -J_{c_-} {}^t B J_{c_0} & * \\ 0 & I_{c_0} & -J_{c_0} {}^t A J_{c_+} \\ 0 & 0 & I_{c_+} \end{pmatrix},$$

we have $g\pi_-(v) = \pi_-(v) - J_{c_-} {}^t B J_{c_0} \pi_-(v)$. Hence

$$gv = g\pi_+(v) + g\pi_-(v) + g\pi_Z(v) = v + A\pi_+(v) - J_{c_-} {}^t B J_{c_0} \pi_-(v).$$

So we can write $f_g(v) = A\pi_+(v) - J_{c_-} {}^t B J_{c_0} \pi_-(v)$.

(ii) For every element $f \in \mathrm{Hom}(V \cap U_{(0)}, U_{(+-)})$, $f(v)$ is written as

$$f(v) = A\pi_+(v) - J_{c_-} {}^t B J_{c_0} \pi_-(v)$$

with some $A \in \mathcal{M}(c_+, c_0; \mathbb{F})$ and $B \in \mathcal{M}(c_0, c_-; \mathbb{F})$. □

Let $\mathcal{F} : V_1 \subset \cdots \subset V_n$ be a full flag in \mathbb{F}^{2n+1} such that $V_n = V$. Define subsets I_1 and I_2 of $I = \{1, \dots, n\}$ by

$$\begin{aligned} I_1 = I_1(\mathcal{F}) &= \{\gamma_1, \dots, \gamma_c\} = \{i \in I \mid V_i \cap U_{(+ -)} \supsetneq V_{i-1} \cap U_{(+ -)}\}, \\ I_2 = I_2(\mathcal{F}) &= \{\delta_1, \dots, \delta_{c_0}\} = \{i \in I \mid V_i \cap U_{(+ -)} = V_{i-1} \cap U_{(+ -)}\} \end{aligned}$$

where $c = c_+ + c_-$ and $\gamma_1 < \cdots < \gamma_c$, $\delta_1 < \cdots < \delta_{c_0}$. Let $\tau(\mathcal{F})$ denote the permutation

$$\tau(\mathcal{F}) : (1 \ 2 \cdots n) \mapsto (\gamma_1 \cdots \gamma_c \delta_1 \cdots \delta_{c_0})$$

of I and $\ell(\tau(\mathcal{F}))$ the inversion number. Then

$$\ell(\tau(\mathcal{F})) = (c_+ + c_-)c_0 - \sum_{i \in I_2(\mathcal{F})} \dim(V_i \cap U_{(+ -)}).$$

Proposition 2.13. (i) Let $\mathcal{F} : V_1 \subset \cdots \subset V_n$ be a full flag in \mathbb{F}^{2n+1} such that $V_n = V$. Then there exists a $g \in N_H$ such that $g\mathcal{F} : gV_1 \subset \cdots \subset gV_n$ is standard.

(ii) Let $\mathcal{F} : V_1 \subset \cdots \subset V_n$ and $\mathcal{F}' : V'_1 \subset \cdots \subset V'_n$ be two standard full flags in \mathbb{F}^{2n+1} . Suppose $\mathcal{F}' = g\mathcal{F}$ with some $g = g_N g_L \in N_H L_H = P_H$. Then $\mathcal{F}' = g_L \mathcal{F}$.

(iii) If $\mathbb{F} = \mathbb{F}_r$, then $|N_H \mathcal{F}| = r^{\ell(\tau(\mathcal{F}))}$.

Proof. (i) Take elements $v_i \in V_i - V_{i-1}$ for $i \in I_2$ and define a subspace $W = \bigoplus_{i \in I_2} \mathbb{F}v_i$ of V . Then we have $V = U_{(+ -)} \oplus W$ and

$$V_i = (V_i \cap U_{(+ -)}) \oplus (V_i \cap W)$$

for $i = 1, \dots, n$.

The subspace W is written as $W = \{v + f(v) \mid v \in V \cap U_{(0)}\}$ with some $f \in \text{Hom}(V \cap U_{(0)}, U_{(+ -)})$. By Lemma 2.12, there exists a $g \in N_H$ such that $gW = V \cap U_{(0)}$. Hence the flag $g\mathcal{F}$ is standard.

(ii) Suppose $V_i = (V_i \cap U_{(+ -)}) \oplus (V_i \cap U_{(0)})$, $V'_i = (V'_i \cap U_{(+ -)}) \oplus (V'_i \cap U_{(0)})$ and $gV_i = V'_i$ with some $g = g_N g_L \in N_H L_H = P_H$. For an element $v \in V_i \cap U_{(0)}$, write $gv = v'_0 + v'_1$ with $v'_0 \in V'_i \cap U_{(+ -)}$ and $v'_1 \in V'_i \cap U_{(0)}$. Since $g_L v = g_N^{-1} gv \in U_{(+ -)} + v'_1$, it follows that $g_L v = v'_1$. Hence

$$g_L(V_i \cap U_{(0)}) = V'_i \cap U_{(0)}.$$

Since $g(V_i \cap U_{(+ -)}) = V'_i \cap U_{(+ -)}$ and since g_N acts trivially on $U_{(+ -)}$, it also follows that $g_L(V_i \cap U_{(+ -)}) = V'_i \cap U_{(+ -)}$.

(iii) We may assume that \mathcal{F} is standard. Let g be an element of N_H . Then

$$g\mathcal{F} = \mathcal{F} \iff f_g(V_i \cap U_{(0)}) \subset V_i \cap U_{(+ -)} \text{ for all } i \in I_2.$$

Take $v_i \in (V_i \cap U_{(0)}) - (V_{i-1} \cap U_{(0)})$ for $i \in I_2$. Then $\{v_i \mid i \in I_2\}$ is a basis of $V \cap U_{(0)}$. The above condition is equivalent to

$$f_g(v_i) \in V_i \cap U_{(+ -)} \text{ for all } i \in I_2.$$

The number of such f_g is $r^{\sum_{i \in I_2} \dim(V_i \cap U_{(+ -)})}$. On the other hand, the number of elements in $\text{Hom}(V \cap U_{(0)}, U_{(+ -)})$ is $r^{(c_+ + c_-)c_0}$. Hence $|N_H \mathcal{F}| = r^{\ell(\tau(\mathcal{F}))}$. \square

2.5. Proof of Theorem 1.2, Theorem 1.4, Proposition 1.7 and Theorem 1.8.

Theorem 1.2 follows from Proposition 2.3 (i), Lemma 2.8 and Proposition 2.11 (i). Proposition 1.7 is proved in Proposition 2.11.

Proof of Theorem 1.4. By Theorem 1.2, we may assume that $t = (U_0, U_d, V)$ ($d = n - a - b$) with $V = V(a, b, c_+, c_-)_{\text{odd}}$, $V = V(a, b, c_+, c_-)_{\text{even}}^0$ or $V = V(a, b, c_+, c_-)_{\text{even}}^1$. Since

$$(2.13) \quad |Gt| = |\{(gU_0, gU_d) \mid g \in G\}| |R_d V| = |M| |PU_d| |R_d V|,$$

we have only to compute $|PU_d|$ and $|R_d V|$. By (2.8), we have

$$(2.14) \quad |PU_d| = r^{d(d+1)/2} \frac{[r]_n}{[r]_d [r]_{n-d}}.$$

On the other hand, it follows from Proposition 2.3 (ii) and (iii) that

$$(2.15) \quad |R_d V| = |Q(R_d \cap L_{W_1})V| = r^{((n-a)(n-a+1)-d(d+1))/2} \frac{[r]_{n-d}}{[r]_a [r]_b} |(R_d \cap L_{W_1})V|.$$

By Proposition 2.11 (iii), we have

$$(2.16) \quad |(R_d \cap L_{W_1})V| = \frac{[r]_d}{[r]_{c_+} [r]_{c_-} [r]_{c_0}} \psi_{c_0}^\varepsilon(r).$$

So the assertion follows from (2.13), (2.14), (2.15) and (2.16). \square

Let U_d and V be as above. Write $H = L_{W_1} \cap R_d \cong \text{GL}_d(\mathbb{F})$. Then $R_d = P \cap P_{U_d}$ is decomposed as $R_d = QH$. By Proposition 2.5 (ii),

$$R(t) = P \cap P_{U_d} \cap P_V = R_d \cap P_V = Q_V(H \cap P_V)$$

where $Q_V = Q \cap P_V$. Define a parabolic subgroup

$$\begin{aligned} P_H &= \{g \in H \mid gU_{(+)} = U_{(+)}, gU_{(-)} = U_{(-)}\} \\ &= \{g \in H \mid gU_{(+)} = U_{(+)}, g(U_{(+)} \oplus U_{(0+)}) = U_{(+)} \oplus U_{(0+)}\} \end{aligned}$$

of H . Then we have a Levi decomposition $P_H = N_H L$ where L is defined in Section 1.3. Since $N_H \subset H \cap P_V \subset P_H$, we can write $H \cap P_V = N_H L_V$ with $L_V = L \cap P_V$. Thus we have:

Lemma 2.14. *We have a bijection*

$$Q_V \times N_H \times L_V \ni (g, h, \ell) \mapsto gh\ell \in R(t) = P \cap P_{U_d} \cap P_V.$$

Proof of Theorem 1.8. (i) Let $\mathcal{F} : V_1 \subset \cdots \subset V_n$ be a full flag in $M_0(V)$. By Proposition 2.5 (i), there exists a $g \in Q_V$ such that

$$gV_i = (gV_i \cap W_0) \oplus (gV_i \cap W_1) \oplus (gV_i \cap W_2),$$

$gV_i \cap W_0 = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_i(\mathcal{F})}$ and that $gV_i \cap W_2 = \mathbb{F}e_{n+d+2} \oplus \cdots \oplus \mathbb{F}e_{n+d+1+b_i(\mathcal{F})}$ for $i = 1, \dots, n$.

Consider the full flag $gV_{\lambda_1} \cap W_1 \subset \cdots \subset gV_{\lambda_d} \cap W_1$ in W_1 . By Proposition 2.13 (i), there exists an $h \in N_H$ such that

$$hgV_i \cap W_1 = (hgV_i \cap (U_{(+)} \oplus U_{(-)})) \oplus (hgV_i \cap U_{(0)}).$$

for $i = 1, \dots, n$. So the flag $hg\mathcal{F}$ is standard.

(ii) Let \mathcal{F} and \mathcal{F}' be two standard full flags in $M_0(V)$ such that $g\mathcal{F} = \mathcal{F}'$ for some $g \in R(t)$. By Lemma 2.14, we can write $g = g_Q g_N g_L$ with $g_Q \in Q_V$, $g_N \in N_H$ and $g_L \in L_V$. Since $g_N g_L \in L_{W_1}$, it follows from Proposition 2.5 (ii) that $g_N g_L \mathcal{F} = \mathcal{F}'$. By Proposition 2.13 (ii), we also have $g_L \mathcal{F} = \mathcal{F}'$ as desired.

(iii) Let $\mathcal{F} : V_1 \subset \cdots \subset V_n$ be a standard full flag in $M_0(V)$. Considering the full flag $V_{\lambda_1} \cap W_1 \subset \cdots \subset V_{\lambda_d} \cap W_1$ in W_1 , we define a permutation

$$\tau' = \tau'(\mathcal{F}) : (\lambda_1 \cdots \lambda_d) \mapsto (\gamma_1 \cdots \gamma_c \delta_1 \cdots \delta_{c_0})$$

of the subset $\{\lambda_1, \dots, \lambda_d\}$ in $\{1, \dots, n\}$. Let $\ell(\tau')$ be the inversion number $\ell(\tau') = |\{(\gamma_i, \delta_j) \mid \gamma_i > \delta_j\}|$. Then we have $\ell(\tau) = \ell(\sigma) + \ell(\tau')$ where

$$\sigma : (1 \ 2 \cdots n) \mapsto (\alpha_1 \cdots \alpha_a \lambda_a \cdots \lambda_d \beta_1 \cdots \beta_b)$$

$$\text{and } \tau : (1 \ 2 \cdots n) \mapsto (\alpha_1 \cdots \alpha_a \gamma_1 \cdots \gamma_c \delta_1 \cdots \delta_{c_0} \beta_1 \cdots \beta_b)$$

are as in Section 2.4 and Section 1.3.

By Proposition 2.13 (ii) and (iii), we have $|N_H L_V \mathcal{F}| = r^{\ell(\tau')} |L_V \mathcal{F}|$. On the other hand, by Proposition 2.5 (ii) and (iii), we have $|Q_V N_H L_V \mathcal{F}| = [r]_a [r]_b r^{\ell(\sigma)} |N_H L_V \mathcal{F}|$. Hence $|R(t)\mathcal{F}| = [r]_a [r]_b r^{\ell(\tau)} |L_V \mathcal{F}|$ by Lemma 2.14. \square

3. ORBITS ON $\mathrm{GL}_n(\mathbb{F})/B$

3.1. Preliminaries. Let \mathbb{F} be an arbitrary field. Let

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{F}^n \quad \text{and} \quad W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset \mathbb{F}^n$$

be two full flags in \mathbb{F}^n . (Write $V_0 = W_0 = \{0\}$ and $V_n = W_n = \mathbb{F}^n$.) Define $d_{i,j} = \dim(V_i \cap W_j)$ for $i, j = 0, 1, 2, \dots, n$ and $c_{i,j} = d_{i,j} - d_{i-1,j} - d_{i,j-1} + d_{i-1,j-1}$ for $i, j = 1, 2, \dots, n$.

Proposition 3.1. *The $n \times n$ matrix $\{c_{i,j}\}$ is a permutation matrix.*

Proof. Since $V_{i-1} \cap W_{j-1} = (V_{i-1} \cap W_j) \cap (V_i \cap W_{j-1})$ and since $V_i \cap W_j \supset (V_{i-1} \cap W_j) + (V_i \cap W_{j-1})$, we have $c_{i,j} \geq 0$. On the other hand, $\sum_{i=1}^n c_{i,j} = d_{n,j} - d_{0,j} - d_{n,j-1} + d_{0,j-1} = j - (j-1) = 1$ and $\sum_{j=1}^n c_{i,j} = d_{i,n} - d_{i,0} - d_{i-1,n} + d_{i,0} = i - (i-1) = 1$. Hence $\{c_{i,j}\}$ is a permutation matrix. \square

We also have the following by the same arguments as above.

Proposition 3.2. *The following four conditions are equivalent:*

- (i) $c_{i,j} = 1$.
- (ii) $d_{i,j} - 1 = d_{i-1,j} = d_{i,j-1} = d_{i-1,j-1}$.
- (iii) $V_i \cap W_j \supsetneq V_{i-1} \cap W_j = V_i \cap W_{j-1} = V_{i-1} \cap W_{j-1}$.
- (iv) $V_i \cap W_j \supsetneq (V_{i-1} \cap W_j) + (V_i \cap W_{j-1})$.

Remark 3.3. (Bruhat decomposition of $G = \mathrm{GL}_n(\mathbb{F})$) Let $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ be the canonical full flag defined by $V_i = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_i$ for $i = 1, \dots, n-1$ with the canonical basis e_1, \dots, e_n of \mathbb{F}^n . The subgroup B of G defined by

$$B = \{g \in G \mid gV_i = V_i \text{ for } i = 1, \dots, n-1\} = \{\text{upper triangular matrices in } G\}$$

is a Borel subgroup of G .

Let $W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ be an arbitrary full flag in \mathbb{F}^n . By Proposition 3.1, we can define a permutation $i = i(j)$ of $\{1, 2, \dots, n\}$ determined by $i = i(j) \iff c_{i,j} = 1$. Suppose $i = i(j)$. Then by Proposition 3.2, we can take vectors $v_i \in V_i \cap W_j$ for $i = 1, \dots, n$ (for $j = 1, \dots, n$) such that

$$v_i \notin V_{i-1} \cap W_j = V_i \cap W_{j-1} = V_{i-1} \cap W_{j-1}.$$

It follows that v_1, \dots, v_n is a basis of \mathbb{F}^n such that $V_i = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_i$ for $i = 1, \dots, n$. It also follows that $v_{i(1)}, \dots, v_{i(n)}$ is a basis of \mathbb{F}^n such that $W_j = \mathbb{F}v_{i(1)} \oplus \cdots \oplus \mathbb{F}v_{i(j)}$ for $j = 1, \dots, n$. Define $n \times n$ matrices

$$g = (v_1 v_2 \cdots v_n) \quad \text{and} \quad w = \{c_{i,j}\} = (e_{i(1)} e_{i(2)} \cdots e_{i(n)}).$$

Then $g \in B$ and $gwV_j = W_j$ for $j = 1, \dots, n$. Thus we have proved $G = \bigsqcup_{w \in \mathcal{W}} BwB$ where \mathcal{W} is the subgroup of G consisting of all the permutation matrices.

3.2. Sp_{2n} -orbits on GL_{2n}/B . Let $\langle \cdot, \cdot \rangle$ denote the alternating form on \mathbb{F}^{2n} defined by

$$\langle e_i, e_j \rangle = \begin{cases} \delta_{i, 2n+1-j} & \text{for } i \leq n, \\ -\delta_{i, 2n+1-j} & \text{for } i > n. \end{cases}$$

Define a subgroup $H = \{g \in G \mid \langle gu, gv \rangle = \langle u, v \rangle \text{ for all } u, v \in \mathbb{F}^{2n}\} \cong \mathrm{Sp}_{2n}(\mathbb{F})$ of $G = \mathrm{GL}_{2n}(\mathbb{F})$. Let $V_1 \subset V_2 \subset \cdots \subset V_{2n-1}$ be a full flag in \mathbb{F}^{2n} . Then there corresponds another (“decreasing”) full flag $V_1^\perp \supset V_2^\perp \supset \cdots \supset V_{2n-1}^\perp$ in \mathbb{F}^{2n} defined by

$$V_i^\perp = \{u \in \mathbb{F}^{2n} \mid \langle u, v \rangle = 0 \text{ for all } v \in V_i\}.$$

Define $d_{i,j} = \dim(V_i \cap V_j^\perp)$ and $c_{i,j} = d_{i,j-1} - d_{i,j} - d_{i-1,j-1} + d_{i-1,j}$. Then we have the following two propositions by Proposition 3.1 and Proposition 3.2.

Proposition 3.4. $\{c_{i,j}\}_{i,j=1}^{2n}$ is a permutation matrix.

Proposition 3.5. The following four conditions are equivalent:

- (i) $c_{i,j} = 1$.
- (ii) $d_{i,j-1} - 1 = d_{i,j} = d_{i-1,j-1} = d_{i-1,j}$.
- (iii) $V_i \cap V_{j-1}^\perp \supsetneq V_i \cap V_j^\perp = V_{i-1} \cap V_{j-1}^\perp = V_{i-1} \cap V_j^\perp$.
- (iv) $V_i \cap V_{j-1}^\perp \supsetneq (V_i \cap V_j^\perp) + (V_{i-1} \cap V_{j-1}^\perp)$

Since the orthogonal space of $V_i \cap V_j^\perp$ is $V_i^\perp + V_j$, we have $\dim(V_i^\perp + V_j) = 2n - d_{i,j}$. So we have

$$\begin{aligned} d_{j,i} &= \dim(V_j \cap V_i^\perp) = \dim V_j + \dim V_i^\perp - \dim(V_j + V_i^\perp) \\ &= j + (2n - i) - (2n - d_{i,j}) = d_{i,j} + j - i \end{aligned}$$

and hence $c_{i,j} = c_{j,i}$.

Lemma 3.6. *Suppose that $c_{i,j} = 1$. Then we have $\langle u, v \rangle \neq 0$ for all $u \in (V_i \cap V_{j-1}^\perp) - (V_{i-1} \cap V_{j-1}^\perp)$ and $v \in (V_j \cap V_{i-1}^\perp) - (V_{j-1} \cap V_{i-1}^\perp)$.*

Proof. Suppose $u \in (V_i \cap V_{j-1}^\perp) - (V_{i-1} \cap V_{j-1}^\perp)$ and $v \in (V_j \cap V_{i-1}^\perp) - (V_{j-1} \cap V_{i-1}^\perp)$. Then

$$\langle u, V_{j-1} \rangle = \{0\} \quad \text{and} \quad V_j = V_{j-1} \oplus \mathbb{F}v.$$

If $\langle u, v \rangle = 0$, then

$$\langle u, V_j \rangle = \langle u, V_{j-1} \oplus \mathbb{F}v \rangle = \langle u, V_{j-1} \rangle = \{0\}$$

and hence $u \in V_i \cap V_j^\perp = V_{i-1} \cap V_{j-1}^\perp$ by Proposition 3.5 (iii). But this contradicts the choice of u . \square

Proof of Proposition 1.9. Suppose $c_{i,i} = 1$. Then we can take an element $v \in (V_i \cap V_{i-1}^\perp) - (V_{i-1} \cap V_{i-1}^\perp)$ by Proposition 3.5 (iii). By Lemma 3.6, we have $\langle v, v \rangle \neq 0$. But this contradicts that $\langle \cdot, \cdot \rangle$ is alternating. \square

Proof of Proposition 1.10. (i) We will prove this by induction on n . Take a pair (i, j) such that $i < j$ and that $c_{i,j} = 1$. By Lemma 3.6, we can take a $v_i \in V_i \cap V_{j-1}^\perp$ and a $v_j \in V_j \cap V_{i-1}^\perp$ such that $\langle v_i, v_j \rangle = 1$. Put $U = \mathbb{F}v_i \oplus \mathbb{F}v_j$. Then we have a direct sum decomposition $\mathbb{F}^{2n} = U \oplus U^\perp$. If $k \leq i - 1$, then $V_k \subset U^\perp$. If $i \leq k \leq j - 1$, then

$$V_k = \mathbb{F}v_i \oplus (V_k \cap U^\perp) = (V_k \cap U) \oplus (V_k \cap U^\perp).$$

If $k \geq j$, then $V_k \supset U$. Hence for every $k = 0, \dots, 2n$, we have $V_k = (V_k \cap U) \oplus (V_k \cap U^\perp)$. We also have $V_\ell^\perp = (V_\ell^\perp \cap U) \oplus (V_\ell^\perp \cap U^\perp)$ and

$$V_k \cap V_\ell^\perp = (V_k \cap V_\ell^\perp \cap U) \oplus (V_k \cap V_\ell^\perp \cap U^\perp)$$

for $k, \ell = 0, \dots, 2n$.

We can consider the full flag $V_1 \cap U^\perp \subset \dots \subset V_{2n-1} \cap U^\perp$ in U^\perp neglecting the two coincidences $V_{i-1} \cap U^\perp = V_i \cap U^\perp$ and $V_{j-1} \cap U^\perp = V_j \cap U^\perp$. Define $d'_{k,\ell} = \dim V_k \cap U_\ell \cap U^\perp$, $d''_{k,\ell} = \dim V_k \cap U_\ell \cap U$, $c'_{k,\ell} = d'_{k,\ell-1} - d'_{k,\ell} - d'_{k-1,\ell-1} + d'_{k-1,\ell}$ and $c''_{k,\ell} = d''_{k,\ell-1} - d''_{k,\ell} - d''_{k-1,\ell-1} + d''_{k-1,\ell}$ for $k, \ell \in I = \{1, \dots, 2n\}$. Then $d_{k,\ell} = d'_{k,\ell} + d''_{k,\ell}$ and $c_{k,\ell} = c'_{k,\ell} + c''_{k,\ell}$. For $k \in I - \{i, j\}$, we see that $V_{k-1} \cap U = V_k \cap U$. So we have $c''_{k,\ell} = 0$ and $c_{k,\ell} = c'_{k,\ell}$ for $k, \ell \in I - \{i, j\}$.

By the assumption of induction, we can take a basis

$$v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_{2n}$$

of U^\perp such that

$$V_k \cap U^\perp = \bigoplus_{\ell \in \{1, \dots, k\} - \{i, j\}} \mathbb{F}v_\ell$$

and that $\langle v_k, v_\ell \rangle = c_{k,\ell}$ for $k, \ell \in I - \{i, j\}$. Thus the basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} satisfies the desired properties. (Remark: We may take $(i, j) = (i_1, j_1) = (1, j_1)$ in the above proof. But we will need such a general argument as above in the proof of Proposition 1.13.)

(ii) Since $\mathbb{F} = \mathbb{F}_r$ consists of r elements, we have $r - 1$ choices of v_1 in $V_1 = V_1 \cap V_{j_1-1}^\perp$. Fix v_1 . Then we have r^{j_1-1} choices of v_{j_1-1} in $V_{j_1} = V_{j_1} \cap V_0^\perp$ such that $\langle v_1, v_{j_1} \rangle = 1$. Write

$$\ell_2 = |\{i \mid 2 < i \text{ and } \sigma(2) > \sigma(i)\}|.$$

Then we have $j_1 = \sigma(2) = \ell_2 + 2$.

Fix v_1 and v_{j_1} . Then the subspaces $U = \mathbb{F}v_1 \oplus \mathbb{F}v_{j_1}$ and U^\perp in (i) are determined. Next we take $v_{i_2} \in V_{i_2} \cap U^\perp \cong \mathbb{F}$. So we have $r - 1$ choices of v_{i_2} . Fix v_{i_2} . We see that

$$\dim(V_{j_2} \cap U^\perp) = \begin{cases} j_2 - 2 & \text{if } j_1 < j_2, \\ j_2 - 1 & \text{if } j_1 > j_2. \end{cases}$$

On the other hand, if we write

$$\ell_4 = |\{i \mid 4 < i \text{ and } \sigma(4) > \sigma(i)\}|,$$

then we have

$$\ell_4 = \begin{cases} j_2 - 4 & \text{if } j_1 < j_2, \\ j_2 - 3 & \text{if } j_1 > j_2. \end{cases}$$

Hence we have r^{ℓ_4+1} choices of $v_{j_2} \in V_{j_2} \cap U^\perp$ such that $\langle v_{i_2}, v_{j_2} \rangle = 1$.

Repeating this procedure, we have $r - 1$ choices of v_{i_k} and $r^{\ell_{2k}+1}$ choices of v_{j_k} if we fix $v_1, v_{j_1}, \dots, v_{i_{k-1}}, v_{j_{k-1}}$. Here

$$\ell_{2k} = |\{i \mid 2k < i \text{ and } \sigma(2k) > \sigma(i)\}|$$

for $k = 1, \dots, n$. Since $\ell(\sigma) = \ell_2 + \ell_4 + \dots + \ell_{2n}$, we have $(r - 1)^n r^{n+\ell(\sigma)}$ choices of the bases v_1, \dots, v_{2n} . \square

3.3. Q_{2n} -orbits on GL_{2n}/B . Let $H \cong Sp_{2n}(\mathbb{F})$ be as in the previous subsection. Let Q_{2n} denote the subgroup of H defined by $Q_{2n} = \{g \in H \mid ge_{2n} = e_{2n}\}$. Then Q_{2n} stabilizes the hyperplane $W = \mathbb{F}e_2 \oplus \dots \oplus \mathbb{F}e_{2n} = (\mathbb{F}e_{2n})^\perp$ in \mathbb{F}^{2n} .

Let $V_1 \subset \dots \subset V_{2n-1}$ be an arbitrary full flag in \mathbb{F}^{2n} . Let S denote the subset of $I \times I$ defined by $S = \{(i, j) \mid V_i \cap V_{j-1}^\perp \not\subset W\}$. Define

$$S_0 = \{(i, j) \in S \mid V_i \cap V_j^\perp \subset W \text{ and } V_{i-1} \cap V_{j-1}^\perp \subset W\}.$$

Lemma 3.7. *Suppose $(i, j) \in S_0$. Then we have:*

- (i) $c_{i,j} = 1$.
- (ii) $V_i \cap V_{j-1}^\perp \cap W = V_{i-1} \cap V_{j-1}^\perp = V_i \cap V_j^\perp = V_{i-1} \cap V_j^\perp$.

Proof. (i) If $(i, j) \in S_0$, then $(V_i \cap V_j^\perp) + (V_{i-1} \cap V_{j-1}^\perp) \subset W$. So we have $(V_i \cap V_j^\perp) + (V_{i-1} \cap V_{j-1}^\perp) \subseteq V_i \cap V_{j-1}^\perp$ and hence $c_{i,j} = 1$ by Proposition 3.5.

(ii) By (i), it follows from Proposition 3.5 that

$$V_i \cap V_{j-1}^\perp \supseteq V_{i-1} \cap V_{j-1}^\perp = V_i \cap V_j^\perp = V_{i-1} \cap V_j^\perp.$$

So the assertion is clear. \square

It is clear that

$$(3.1) \quad (i, j) \in S \text{ and } i \leq i', j \geq j' \implies (i', j') \in S.$$

So the subset S_0 determines S by

$$(i, j) \in S \iff i \geq i_0 \text{ and } j \leq j_0 \text{ for some } (i_0, j_0) \in S_0.$$

For example, if $n = 2$ and $S_0 = \{(2, 1), (4, 3)\}$, then

$$S = \{(2, 1), (3, 1), (4, 1), (4, 2), (4, 3)\}.$$

Proof of Proposition 1.12. By Lemma 3.7, we may assume that $x_1 < x_2 < \dots < x_s$. If $i < j$ and $y_j < y_i$, then it follows from $(x_i, y_i) \in S_0$ that $(x_j, y_j + 1) \in S$ by (3.1). But this contradicts that $(x_j, y_j) \in S_0$. Thus we have

$$y_1 \leq \dots \leq y_s.$$

By Lemma 3.7, the numbers $x_1, \dots, x_s, y_1, \dots, y_s$ are distinct. \square

Write $I_{(A)} = I - \{x_1, \dots, x_s, y_1, \dots, y_s\}$.

Proof of Proposition 1.13. (i) We will prove this by induction on n . First assume that $s < n$. Then we can take a pair (i, j) in $I_{(A)}$ such that $i < j$ and that $c_{i,j} = 1$. By Lemma 3.6, we can take a $v_i \in V_i \cap V_{j-1}^\perp$ and a $v_j \in V_j \cap V_{i-1}^\perp$ such that $\langle v_i, v_j \rangle = 1$. If $(i, j) \notin S$, then

$$v_i \in V_i \cap V_{j-1}^\perp \subset W.$$

On the other hand, if $(i, j) \in S$, then there exists an $(x_t, y_t) \in S_0$ such that $i > x_t$ and $j < y_t$. By Lemma 3.7, there exists a $v \in V_{x_t} \cap V_{y_t-1}$ such that $v \notin W$. Since $x_t < i$, we have $\langle v, v_j \rangle = 0$. If $v_i \notin W$, then we can replace v_i by $v_i + \alpha v \in V_i \cap V_{j-1}^\perp \cap W$ with some $\alpha \in \mathbb{F}^\times$ since $V_{x_t} \subset V_i$ and $V_{y_t}^\perp \subset V_{j-1}^\perp$. Thus we may assume that $v_i \in W$. In the same way, we may also assume that $v_j \in W$.

Put $U = \mathbb{F}v_i \oplus \mathbb{F}v_j$ and consider the direct sum decomposition $\mathbb{F}^{2n} = U \oplus U^\perp$ as in the proof of Proposition 1.10. Since $U \subset W$, e_{2n} is contained in U^\perp . So we may assume that we have chosen a basis

$$v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_{2n}$$

of U^\perp satisfying the conditions (a), (b) and (c) for U^\perp by the assumption of induction. Then the basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} is a desired one.

Finally we consider the case of $s = n$. By Proposition 1.10, we can take a basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} such that $V_i = \mathbb{F}v_1 \oplus \dots \oplus \mathbb{F}v_i$ for $i = 1, \dots, 2n$ and that $\langle v_i, v_j \rangle = c_{i,j}$ for $i < j$. By Lemma 3.7, v_{x_1}, \dots, v_{x_s} are not contained in W . So we can normalize them so that

$$\langle v_{x_1}, e_{2n} \rangle = \dots = \langle v_{x_s}, e_{2n} \rangle = 1.$$

We can also normalize v_{y_1}, \dots, v_{y_s} so that $\langle v_{x_t}, v_{y_t} \rangle = \varepsilon_t$ where

$$\varepsilon_t = \begin{cases} 1 & \text{if } x_t < y_t, \\ -1 & \text{if } x_t > y_t. \end{cases}$$

If $y_t < x_t$, then $(y_t, x_t) \notin S$. So we have $v_{y_t} \in W$. On the other hand, if $y_t > x_t$ and $v_{y_t} \notin W$, then we can replace v_{y_t} by $v_{y_t} + \alpha v_{x_t} \in W$ with some $\alpha \in \mathbb{F}^\times$. Thus the basis v_1, \dots, v_{2n} of \mathbb{F}^{2n} satisfies the desired properties.

(ii) By Proposition 1.10, we have $(r-1)^{nr^{n+\ell(\sigma)}}$ choices of the bases without the condition (c). By the first condition

$$v_i \in W \text{ for } i \neq x_1, \dots, x_s$$

in (c), the number of choices of v_i becomes $1/r$ of the number without the condition (c) for each i such that

$$(3.2) \quad (i, j) \notin S - S_0 \text{ with } c_{i,j} = 1$$

as in the proof of (i). Since there are $m = m(\mathcal{F})$ indices i satisfying the condition (3.2) by the definition, we divide the number by r^m . On the other hand, the second condition

$$\langle v_{x_1}, e_{2n} \rangle = \dots = \langle v_{x_s}, e_{2n} \rangle = 1$$

in (c) is the condition on the length of vectors v_{x_1}, \dots, v_{x_s} . So we divide the number by $(r-1)^s$. Thus we have $(r-1)^{n-s} r^{n+\ell(\sigma)-m}$ choices of the bases. \square

Proof of Theorem 1.14. (i) For each partition $I = I_{(A)} \sqcup I_{(X)} \sqcup I_{(Y)}$ and each $\{c_{i,j}\} \in C(I_{(A)})$, we construct a “standard” basis u_1, \dots, u_{2n} of \mathbb{F}^{2n} as follows. We can take a unique subsequence $i_1 < \dots < i_{n-s}$ in $I_{(A)}$ such that $c_{i_1, j_1} = \dots = c_{i_{n-s}, j_{n-s}} = 1$ with some $j_1, \dots, j_{n-s} \in I_{(A)}$ and that $i_t < j_t$ for $t = 1, \dots, n-s$. Define

$$\begin{aligned} u_{i_1} &= e_{s+1}, \dots, u_{j_{n-s}} = e_n, \\ u_{j_1} &= e_{2n-s}, \dots, u_{j_{n-s}} = e_{n+1}, \\ u_{x_1} &= e_1 + e_2, u_{x_2} = e_1 + e_3, \dots, u_{x_{s-1}} = e_1 + e_s, u_{x_s} = e_1, \\ u_{y_1} &= \varepsilon_1 e_{2n-1}, u_{y_2} = \varepsilon_2 e_{2n-2}, \dots, u_{y_{s-1}} = \varepsilon_{s-1} e_{2n-s+1}, \\ u_{y_s} &= \varepsilon_s (e_{2n} - e_{2n-1} - \dots - e_{2n-s+1}). \end{aligned}$$

Then the basis vectors u_1, \dots, u_{2n} satisfy the properties:

$$(3.3) \quad \langle u_i, u_j \rangle = c_{i,j} \text{ for } i < j,$$

$$(3.4) \quad u_i \in W \text{ for } i \neq x_1, \dots, x_s$$

and

$$(3.5) \quad \langle u_{x_1}, e_{2n} \rangle = \dots = \langle u_{x_s}, e_{2n} \rangle = 1.$$

Let v_1, \dots, v_{2n} be the basis of \mathbb{F}^{2n} given in Proposition 1.13 (i). Let g be the element of $\text{GL}_{2n}(\mathbb{F})$ defined by $gv_i = u_i$ for $i = 1, \dots, 2n$. By (3.3) and Proposition 1.13 (i) (b), g is an element of $H \cong \text{Sp}_{2n}(\mathbb{F})$. By (3.4), (3.5) and Proposition 1.13 (i) (c), we have

$$\begin{aligned} W &= \mathbb{F}(u_{x_1} - u_{x_2}) \oplus \dots \oplus \mathbb{F}(u_{x_{s-1}} - u_{x_s}) \oplus \bigoplus_{i \notin \{x_1, \dots, x_s\}} \mathbb{F}u_i \\ &= \mathbb{F}(v_{x_1} - v_{x_2}) \oplus \dots \oplus \mathbb{F}(v_{x_{s-1}} - v_{x_s}) \oplus \bigoplus_{i \notin \{x_1, \dots, x_s\}} \mathbb{F}v_i. \end{aligned}$$

So the element g stabilizes W . Hence we have $ge_{2n} = \beta e_{2n}$ with some $\beta \in \mathbb{F}^\times$. Moreover we have

$$\beta = \langle u_{x_1}, \beta e_{2n} \rangle = \langle gv_{x_1}, ge_{2n} \rangle = \langle v_{x_1}, e_{2n} \rangle = 1.$$

Thus we have proved $g \in Q_{2n}$. Since the flag $gV_1 \subset \cdots \subset gV_n$ is written as $gV_i = \mathbb{F}u_1 \oplus \cdots \oplus \mathbb{F}u_i$ ($i = 1, \dots, 2n-1$) using the standard basis u_1, \dots, u_n , we have proved (i).

(ii) is clear and (iii) follows from Proposition 1.13 (ii). \square

3.4. Proof of Theorem 1.19. *Proof of Lemma 1.18.* Let $\xi(2k, s)$ be the number of the words with $2k$ letters consisting of

$$a, b, \dots, A, B, \dots, X, Y$$

with $|I_X| = |I_Y| = s$. There are $k-s$ pairs of a, b, \dots , or A, B, \dots and we can assign a signature $+1$ or -1 to each pair according as they are small or capital. So we have

$$\xi(2k, s) = 2^{k-s}(2k-2s-1)(2k-2s-3) \cdots 1 \frac{(2k)!}{(s!)^2(2k-2s)!} = \frac{(2k)!}{(s!)^2(k-s)!}.$$

Thus we have the desired formula for $\xi(2k) = \sum_{s=0}^k \xi(2k, s)$. We can prove $\xi(2k-1) = \sum_{s=1}^k (2k-1)!/s!(s-1)!(k-s)!$ in the same way. \square

Proof of Theorem 1.19. (i) We can divide every word w into the subword w_1 consisting of $\alpha, \beta, +, -$ and the subword w_2 consisting of the other letters. There are 4^{n-k} choices of w_1 and $\xi(k)$ choices of w_2 if the length of w_2 is k . Considering the number of partitions of w into two subwords with $n-k$ letters and k letters, we get the desired formula

$$|\Delta G \setminus M \times M \times M_0| = \sum_{k=0}^n 4^{n-k} \binom{n}{k} \xi(k).$$

The proof of (ii) is similar because we have only to consider words without α and β . \square

3.5. Proof of Theorem 1.20. *Proof of Theorem 1.20.* (i) is proved easily.

(ii) By the same arguments as in the proof of Proposition 2.3 (iii), we have $|(N_{W_0} \cap G')V| = r^{bd+b(b-1)/2} = r^{((n-a)(n-a-1)-d(d-1))/2}$ and hence

$$(3.6) \quad |(Q \cap G')V| = r^{((n-a)(n-a-1)-d(d-1))/2} \frac{[r]_{n-d}}{[r]_a[r]_b}.$$

Noting that $R_d \cap L_{W_1} \subset G'$, we have

$$|(R_d \cap G')V| = |(Q \cap G')(R_d \cap L_{W_1})V| = r^{((n-a)(n-a-1)-d(d-1))/2} \frac{[r]_{n-d}}{[r]_a[r]_b} |(R_d \cap L_{W_1})V|.$$

On the other hand, we also have $|(P \cap G')U_d| = r^{d(d-1)/2} [r]_n / ([r]_d [r]_{n-d})$ by (3.6). Combining with (2.16), we get the desired formula

$$|G't| = |M^0| |(P \cap G')U_d| |(R_d \cap G')V| = |M^0| \frac{r^{(n-a)(n-a-1)/2} [r]_n \psi_{c_0}^0(r)}{[r]_a [r]_b [r]_{c_+} [r]_{c_0} [r]_{c_-}}. \quad \square$$

4. APPENDIX

Let \mathbb{F} be an arbitrary field and let $\mathbb{F}^n = U_+ \oplus U_-$ be the direct sum decomposition of \mathbb{F}^n with $U_+ = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{m_+}$ and $U_- = \mathbb{F}e_{m_++1} \oplus \cdots \oplus \mathbb{F}e_n$ ($n = m_+ + m_-$). Let H denote the subgroup of $G = \mathrm{GL}_n(\mathbb{F})$ defined by

$$H = \{g \in G \mid gU_+ = U_+, gU_- = U_-\}.$$

In this appendix, we will give a proof of the H -orbit decomposition on the full flag variety of $G = \mathrm{GL}_n(\mathbb{F})$.

Let $\pi_+ : \mathbb{F}^n \rightarrow U_+$ and $\pi_- : \mathbb{F}^n \rightarrow U_-$ denote the projections with respect to the direct sum decomposition $\mathbb{F}^n = U_+ \oplus U_-$. For a full flag $\mathcal{F} : V_1 \subset \cdots \subset V_{n-1}$ in \mathbb{F}^n , define

$$d_{i,j}^+ = \dim(\pi_+(V_i) \cap V_j), \quad d_{i,j}^- = \dim(\pi_-(V_j) \cap V_i)$$

for $i, j = 0, \dots, n$ and

$$c_{i,j}^\pm = d_{i,j}^\pm - d_{i,j-1}^\pm - d_{i-1,j}^\pm + d_{i-1,j-1}^\pm, \quad c_{i,j} = c_{i,j}^+ + c_{i,j}^-$$

for $i, j = 1, \dots, n$.

Lemma 4.1. (i) $\{c_{i,j}\}$ is a permutation matrix.

(ii) $i > j \implies c_{i,j}^+ = 0$ and $i < j \implies c_{i,j}^- = 0$.

(iii) $c_{i,j} = c_{j,i}$.

Proof. (i) As in the proof of Proposition 3.1, we have $c_{i,j}^+ \geq 0$ and $c_{i,j}^- \geq 0$. On the other hand, we have

$$\begin{aligned} \sum_{j=1}^n c_{i,j}^+ &= d_{i,n}^+ - d_{i,0}^+ + d_{i-1,n}^+ - d_{i-1,0}^+ = \dim \pi_+(V_i) - \dim \pi_+(V_{i-1}) \\ \text{and } \sum_{j=1}^n c_{i,j}^- &= d_{i,n}^- - d_{i,0}^- + d_{i-1,n}^- - d_{i-1,0}^- = \dim(V_i \cap U_-) - \dim(V_{i-1} \cap U_-). \end{aligned}$$

Hence $\sum_{j=1}^n c_{i,j} = \dim V_i - \dim V_{i-1} = 1$. In the same way, we also have $\sum_{i=1}^n c_{i,j} = 1$. So the matrix $\{c_{i,j}\}$ is a permutation matrix.

(ii) If $i > j$, then $\pi_+(V_{i-1}) \cap V_j = \pi_+(V_{i-1}) \cap V_j \cap U_+ = V_j \cap U_+$. In the same way, we have $\pi_+(V_i) \cap V_j = V_j \cap U_+$. Hence $c_{i,j}^+ = 0$. The second formula is similar.

(iii) Suppose $i < j$. Then it follows from (ii) that

$$\begin{aligned} c_{i,j} &= c_{i,j}^+ = -\dim(\pi_+(V_i) + V_j) + \dim(\pi_+(V_i) + V_{j-1}) \\ &\quad + \dim(\pi_+(V_{i-1}) + V_j) - \dim(\pi_+(V_{i-1}) + V_{j-1}) \end{aligned}$$

since $d_{i,j}^+ = \dim \pi_+(V_i) + \dim V_j - \dim(\pi_+(V_i) + V_j)$ for $i, j = 0, \dots, n$. It also follows from (ii) that

$$\begin{aligned} c_{j,i} &= c_{j,i}^- = -\dim(\pi_-(V_i) + V_j) + \dim(\pi_-(V_i) + V_{j-1}) \\ &\quad + \dim(\pi_-(V_{i-1}) + V_j) - \dim(\pi_-(V_{i-1}) + V_{j-1}). \end{aligned}$$

Since $\pi_+(V_k) + V_\ell = \pi_-(V_k) + V_\ell$ for $k \leq \ell$, we have $c_{i,j} = c_{j,i}$. \square

Define subsets

$$\begin{aligned} I_{(+)} &= \{k_1^+, \dots, k_{m_+-s}^+\} = \{i \in I \mid c_{i,i}^+ = 1\}, \\ I_{(-)} &= \{k_1^-, \dots, k_{m_--s}^-\} = \{i \in I \mid c_{i,i}^- = 1\}, \\ I_{(1)} &= \{i_1, \dots, i_s\} = \{i \in I \mid c_{i,j} = 1 \text{ for some } j > i\}, \\ I_{(2)} &= \{j_1, \dots, j_s\} = \{j \in I \mid c_{i,j} = 1 \text{ for some } i < j\} \end{aligned}$$

of $I = \{1, \dots, n\}$ with $k_1^+ < \dots < k_{m_+-s}^+$, $k_1^- < \dots < k_{m_--s}^-$, $j_1 < \dots < j_s$ and

$$c_{i_t, j_t} = 1 \text{ for } t = 1, \dots, s.$$

Then $I = I_{(+)} \sqcup I_{(-)} \sqcup I_{(1)} \sqcup I_{(2)}$. Write $I_{(+)} \sqcup I_{(-)} = \{k_1, \dots, k_{n-2s}\}$ with $k_1 < \dots < k_{n-2s}$. Define a permutation

$$\sigma : (1 \ 2 \ \dots \ n) \mapsto (i_s \ \dots \ i_1 \ k_1 \ \dots \ k_{n-2s} \ j_1 \ \dots \ j_s)$$

of I and the inversion number $\ell(\sigma)$. (Remark: Let τ denote the permutation corresponding to the matrix $\{c_{i,j}\}$. Then we can prove $\ell(\tau) = s(n-s) - 2\ell(\sigma)$.)

Proposition 4.2. *For every full flag $\mathcal{F} : V_1 \subset \dots \subset V_{n-1}$ in \mathbb{F}^n , define $c_{i,j} = c_{i,j}^+ + c_{i,j}^-$ as above.*

- (i) *We can take a basis v_1, \dots, v_n of \mathbb{F}^n such that*
 - (a) $V_i = \mathbb{F}v_1 \oplus \dots \oplus \mathbb{F}v_i$ for $i = 1, \dots, n$.
 - (b) $c_{i,i}^+ = 1 \implies v_i \in U_+$, $c_{i,i}^- = 1 \implies v_i \in U_-$ and

$$i < j, \ c_{i,j} = 1 \implies v_i \notin U_+ \cup U_-, \ v_j = \pi_+(v_i).$$

- (ii) *Define a basis u_1, \dots, u_n of \mathbb{F}^n by*

$$\begin{aligned} u_{k_t^+} &= e_t && \text{for } t = 1, \dots, m_+ - s, \\ u_{k_t^-} &= e_{m_++t} && \text{for } t = 1, \dots, m_- - s, \\ u_{i_t} &= e_{m_+-s+t} + e_{n-s+t} && \text{for } t = 1, \dots, s, \\ u_{j_t} &= e_{m_+-s+t} && \text{for } t = 1, \dots, s \end{aligned}$$

and define $g \in G$ by $gv_i = u_i$ for $i = 1, \dots, n$. Then $g \in H$.

- (iii) *If $\mathbb{F} = \mathbb{F}_r$, then the number $\mathcal{N}(\mathcal{F})$ of the bases satisfying the conditions in (i) is*

$$(r-1)^{n-s} r^{((m_+-s)(m_+-s-1)+(m_--s)(m_--s-1))/2 + \ell(\sigma)}.$$

Proof. (i) Suppose $c_{i,j}^+ = 1$. Then we can take a $v \in \pi_+(V_i) \cap V_j$ such that $v \notin \pi_+(V_i) \cap V_{j-1} = \pi_+(V_{i-1}) \cap V_j$. If $i = j$, then $v_i = v \in U_+$ satisfies $V_i = V_{i-1} \oplus \mathbb{F}v_i$. Suppose $i < j$ and take a $v_i \in V_i$ so that $\pi_+(v_i) = v$. Then $V_i = V_{i-1} \oplus \mathbb{F}v_i$ and $v_i \notin U_-$. If $v_i \in U_+$, then $v = v_i \in V_i \subset V_{j-1}$ a contradiction. Hence $v_i \notin U_+$. Take $v_j = v = \pi_+(v_i)$. Then $V_j = V_{j-1} \oplus \mathbb{F}v_j$.

If $c_{i,i}^- = 1$, then we take a $v_i \in \pi_-(V_i) \cap V_i$ such that $v_i \notin \pi_-(V_i) \cap V_{i-1} = \pi_-(V_{i-1}) \cap V_i$. Then $v_i \in U_-$ and $V_i = V_{i-1} \oplus \mathbb{F}v_i$.

- (ii) is clear from (i).

(iii) We will prove this by induction on n . First suppose $c_{n,n}^+ = 1$. Then $V_{n-1} = (V_{n-1} \cap U_+) \oplus U_-$ and so the number of the bases v_1, \dots, v_{n-1} of V_{n-1} satisfying the conditions in (i) is

$$(r-1)^{n-s-1} r^{((m_+-s-1)(m_+-s-2)+(m_--s)(m_--s-1))/2+\ell(\sigma')}.$$

by the assumption of induction. Here σ' is the permutation

$$\sigma' : (1\ 2 \cdots n-1) \mapsto (i_s \cdots i_1 k_1 \cdots k_{n-2s-1} j_1 \cdots j_s).$$

For each basis v_1, \dots, v_{n-1} of V_{n-1} , we have $(r-1)r^{m_+-1}$ choices of $v_n \in U_+ - (U_+ \cap V_{n-1})$. Since

$$(m_+ - s)(m_+ - s - 1)/2 = (m_+ - s - 1)(m_+ - s - 2)/2 + (m_+ - s - 1)$$

and since $\ell(\sigma) = \ell(\sigma') + s$, we have

$$\mathcal{N}(\mathcal{F}) = (r-1)^{n-s} r^{((m_+-s)(m_+-s-1)+(m_--s)(m_--s-1))/2+\ell(\sigma)}.$$

The case of $c_{n,n}^- = 1$ is similar.

So we may assume $c_{p,n} = c_{p,n}^+ = 1$ with some $p < n$. Consider the subspace $W = (V_{n-1} \cap U_+) \oplus (V_{n-1} \cap U_-)$ of \mathbb{F}^n . Then the number of the bases $v_1, \dots, v_{p-1}, v_{p+1}, \dots, v_{n-1}$ of W satisfying the conditions in (i) is

$$(r-1)^{n-s-1} r^{((m_+-s)(m_+-s-1)+(m_--s)(m_--s-1))/2+\ell(\sigma')}$$

by the assumption of induction where σ' is the permutation

$$\sigma' : (1\ 2 \cdots p-1\ p+1 \cdots n-1) \mapsto (i_{s-1} \cdots i_1 k_1 \cdots k_{n-2s} j_1 \cdots j_{s-1})$$

of $\{1 \dots, p-1, p+1, \dots, n-1\}$. (Note that $p = i_s$ and $n = j_s$.) For each basis $v_1, \dots, v_{p-1}, v_{p+1}, \dots, v_{n-1}$ of W , we have $(r-1)r^{p-1}$ choices of $v_p \in V_p - V_{p-1}$. Since $\ell(\sigma) = \ell(\sigma') + p - 1$, we have the desired formula for $\mathcal{N}(\mathcal{F})$ noting that $v_n = \pi_+(v_p)$. \square

By this proposition, we can express orbits by “+−ab-symbols” as in Fig.1 (c.f. [MO90]). We can easily count the number of orbits:

Corollary 4.3. $|H \backslash M| = \sum_{s=0}^{\min(m_+, m_-)} \frac{n!}{2^s s! (m_+ - s)! (m_- - s)!}.$

Corollary 4.4. For each full flag \mathcal{F} in \mathbb{F}^n , define s and σ as above. If $\mathbb{F} = \mathbb{F}_r$, then

$$|H\mathcal{F}| = (r-1)^s r^{s(n-s-1)-\ell(\sigma)} [r]_{m_+} [r]_{m_-}.$$

Proof. Note that

$$\begin{aligned} |H| &= (r^{m_+} - 1)(r^{m_+} - r) \cdots (r^{m_+} - r^{m_+-1})(r^{m_-} - 1)(r^{m_-} - r) \cdots (r^{m_-} - r^{m_--1}) \\ &= (r-1)^n r^{(m_+(m_+-1)+m_-(m_--1))/2} [r]_{m_+} [r]_{m_-}. \end{aligned}$$

Since $(m_+(m_+-1)+m_-(m_--1))/2 - ((m_+-s)(m_+-s-1)+(m_--s)(m_--s-1))/2 = s(n-s-1)$, we have

$$|H\mathcal{F}| = |H|/\mathcal{N}(\mathcal{F}) = (r-1)^s r^{s(n-s-1)-\ell(\sigma)} [r]_{m_+} [r]_{m_-}.$$

\square

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